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DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING
COLLEGE OF ENGINEERING AND TECHNOLOGY
OLD DOMINION UNIVERSITY
NORFOLK, VIRGINIA 23529

A NEW METHOD FOR RECOGNIZING QUADRIC SURFACES FROM RANGE DATA AND ITS APPLICATION TO TELEROBOTICS AND AUTOMATION

By

Nicolas Alvertos, Principal Investigator

and

Ivan D'Cunha, Graduate Research Assistant

Progress Report
For the period ended February 29, 1992

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National Aeronautics and Space Administration
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Hampton, Virginia 23665

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A New Method for Recognizing Quadric Surfaces from Range Data and Its Application to Telerobotics and Automation (phase 1)

by

Nicolas Alvertos* and Ivan D'Cunha**

Abstract

In this phase of the proposed research a feature set of two-dimensional curves is obtained after intersecting symmetric objects like spheres, cones, cylinders, ellipsoids, paraboloids, and parallelepipeds with two planes. After determining the location and orientation of the objects in space, these objects are aligned so as to lie on a plane parallel to a suitable coordinate system. These objects are then intersected with a horizontal and a vertical plane. Experiments were carried out with range images of sphere and cylinder. The 3-D discriminant approach has been utilized to recognize quadric surfaces made up of simulated data. Its application to real data has also been investigated.

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INTRODUCTION

Three-Dimensional object characterization and recognition is a primary task in the field of computer vision and automation. With the rapid improvement in quality and accuracy of several active sensors, digitized range data has proved to be the prime candidate utilized for surface characterization. Several techniques have and are being developed, most of which rely heavily on differential geometry of smooth surfaces to describe 3-D objects. Although occlusion can be detected more easily in range images than in intensity based images, researchers have been trying hard to solve the problems of range image segmentation accurately and efficiently. Segmentation has been achieved for simulated and real range data for groups of symmetric objects of spheres, ellipsoids, cylinders and parallelepipeds.

In this research we extend upon our earlier proposed research [1] wherein quadric surfaces are characterized by a feature set of two-dimensional curves. These curves are obtained after intersecting several 3-D surfaces with planes in various orientations. We put forward a new technique for determining the position and orientation of objects with respect to a fixed coordinate system. An efficient implementation of this algorithm will aid in obtaining a unique set of feature vectors for the various 3-D objects considered and thereby aid in the recognition process. The 3-D discriminant approach of classification and reduction of quadrics which was discussed in our earlier research [1,2], has been implemented and tested for simulated as well as synthetic range data.

PREVIOUS WORK

Since real range data obtained is not noise free, median filtering is performed [3] and the resulting images then undergo a series of curvature analysis tests to validate which image data can best be used for the recognition process. Best fit plots [2] also justifies the results obtained from curvature analysis as to which set of coefficients best fits the range data.

A quadric surface as shown in equation (1) is represented as a second degree polynomial,

$$F(x,y,z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2px + 2qy + 2rz + d = 0 \qquad (1)$$

consisting of ten coefficients, which are obtained using the approach [4] developed by Groshong and Bilbro.

Our proposed research has been applied earlier [1] to symmetric objects such as spheres, ellipsoids, cones, cylinders and parallelepiped. In the present research we extend it to various other quadric surfaces such as parabolic and hyperbolic cylinders, elliptic and parabolic paraboloids, and hyperboloids of one and two sheets. Each of these surfaces have been characterized separately in terms of its intersections with various planes.

SURFACE CHARACTERIZATION

We will consider the representation and intersections (by two planes) of the following : parabolic and hyperbolic cylinders, elliptic and parabolic paraboloids, and hyperboloids of one and two sheets. Figure 1 illustrates each of the quadric surfaces to be used for the recognition process.

Assume that all the objects are resting on planes parallel to the yz plane and that their axis of rotation is parallel to any one axis of the coordinate axis. Then in relation to equation (1) the product terms, namely yz, xz, and xy will be missing. We will refer to plane 1 as the plane that intersects the object parallel to the yz-axis, i.e., x = constant. Also let us refer to plane 2 as the plane that intersects the object parallel to the xz-axis, i.e., y = constant.

Let us initially consider the parabolic cylinder. This surface is symmetrical with respect to the xz plane the xy plane and the x-axis, and is a cylinder parallel to the y-

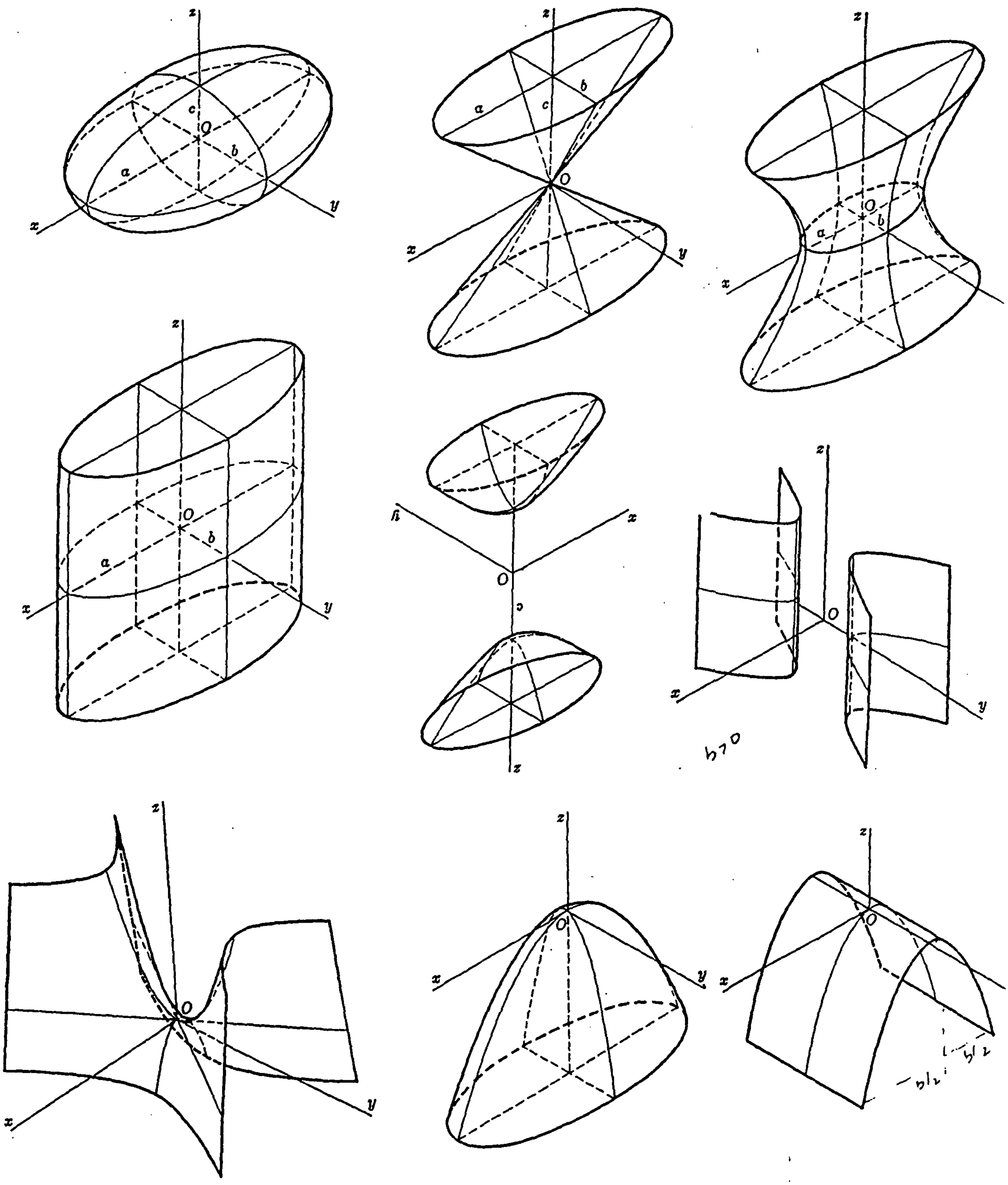


Figure 1. Quadric surfaces from left to right and top to bottom: Quadric cone, Hyperboloid of one sheet, Elliptic cylinder, Hyperboloid of two sheets, Hyperbolic cylinder, Hyperbolic paraboloid, Elliptic paraboloid, and Parabolic cylinder.

axis. Consider the cylinder lying on a plane parallel to the yz plane, thereby it is parallel to the y axis. The quadric representation without the product terms reduces to

$$f(x,y,z) = az^2 + 2px + 2rz + d = 0 \tag{2}$$

Upon completing squares it reduces to

$$(z + \frac{r}{a})^2 + \frac{2px}{a} = 0$$

only if $d = -\frac{r^2}{a}$.

Intersection of the parabolic cylinder with plane 1, i.e., x = k, 0 < k < 2p/ab, where b is any finite positive quantity, signifying the width of the base of the cylinder, yields

$$(z + \frac{r}{a})^2 = -\frac{2pk}{a}$$

which when solved results in a pair of parallel lines.

Consider the intersection of the parabolic cylinder with the plane 2, i.e., y = k. Since equation (2) is independent of the variable y, the resultant curve intercepted is the same as equation (1), which is a equation of a parabola.

The hyperbolic cylinder is considered next. This surface is symmetrical with respect to the coordinate planes and axes and any point on the z-axis. Since this cylinder is parallel to the x-axis, this variable will be missing in its representation. Equation (1) reduces to

$$ay^2 + bz^2 + 2qy + 2rz + d = 0 \tag{3}$$

where ab < 0. Completing squares

$$\frac{(y + \frac{q}{a})^2}{\frac{1}{a}} - \frac{(z + \frac{r}{b})^2}{-\frac{1}{b}} - 1 = 0$$

only if $d = \frac{q^2}{a} + \frac{r^2}{b} - 1$. Also -1/b is a positive quantity. Intersection of the cylinder with plane 1, i.e., x = k, would generate a hyperbola. Since equation (3) is independent of the variable x, the curve intercepted is the one represented by equation (3). Intersection of the hyperbolic cylinder with plane 2, i.e., y = k would result in the equation

$$\frac{(z + \frac{r}{b})^2}{-\frac{1}{b}} = \frac{(k + \frac{q}{a})^2}{\frac{1}{a}} + 1$$

which when solved results in a pair of straight lines.

Let us consider the elliptic paraboloid next. This surface is symmetrical with respect to the xz plane, the xy plane and the x-axis. It is represented as

$$ay^2 + bz^2 + 2px + 2qy + 2rz + d = 0 \tag{4}$$

Equation (4) upon completing squares, reduces to the form

$$\frac{(y + \frac{q}{a})^2}{\frac{1}{a}} + \frac{(z + \frac{r}{b})^2}{\frac{1}{b}} + \frac{x}{\frac{1}{2p}} = 0$$

only if $d = \frac{r^2}{b} + \frac{q^2}{a}$.

$\sqrt{\frac{1}{a}}, \sqrt{\frac{1}{b}}$ are the semi major and minor axes of the paraboloid, whereas 1/2p is the height of the paraboloid.

Consider the intersection of the elliptic paraboloid with the plane 1, i.e., x = k, where 0 < k < 1/2r. Under these circumstances, the above equation results to

$$\frac{(z + \frac{r}{b})^2}{\frac{1}{b}} + \frac{(y + \frac{q}{a})^2}{\frac{1}{a}} = \frac{-k}{\frac{1}{2p}} \tag{5}$$

where $\dfrac{-k}{\frac{1}{2p}}$ is a positive quantity. Equation (5) is that of an ellipse.

Consider the intersection of the surface with the plane 2, i.e., y = k, where $\frac{-q}{a} - \sqrt{\frac{1}{a}} < k < \frac{-q}{a} + \sqrt{\frac{1}{a}}$. Then

$$\frac{(z + \frac{r}{b})^2}{\frac{1}{b}} = -\left[\frac{x}{\frac{1}{2p}} - \frac{(k + \frac{q}{a})^2}{\frac{1}{a}}\right]$$

which is an equation of a parabola.

The next 3-D surface considered is a hyperbolic paraboloid. Unlike the elliptic paraboloid, this object is symmetrical with respect to the xz plane, the xy plane and the x-axis. Its representation is as shown below :

$$ay^2 + bz^2 + 2px + 2qy + 2rz + d = 0$$

However in this case $ab < 0$. Upon completing squares we have

$$\frac{(y + \frac{q}{a})^2}{\frac{-1}{a}} + \frac{(z + \frac{r}{b})^2}{\frac{1}{b}} + \frac{x}{\frac{1}{2p}} = 0$$

only if $d = \frac{r^2}{b} + \frac{q^2}{a}$.

Intersecting the surface with the plane 1, i.e, x = k, results in

$$\frac{(y + \frac{q}{a})^2}{\frac{-1}{a}} + \frac{(z + \frac{r}{b})^2}{\frac{1}{b}} = \frac{-k}{\frac{1}{2p}}$$

where $-1/a$ is a positive quantity. This equation is that of a hyperbola. In the case when $x = k = 0$, it will result in a pair of parallel lines which is a degenerate case of a hyperbola.

Consider the case when the object is intersected with plane 2, i.e., y = k, then

$$\frac{(z + \frac{r}{b})^2}{\frac{1}{b}} = -\frac{(k + \frac{q}{a})^2}{\frac{1}{a}} - 2px$$

which is an equation of a parabola.

The object considered next is a hyperboloid of one sheet. This surface is symmetrical with respect to each coordinate plane and axis and the origin. The equation of a hyperboloid lying in a plane parallel to yz plane is

$$ax^2 + by^2 + bz^2 + 2px + 2qy + 2rz + d = 0 \tag{6}$$

where we assumed the base to be a circular one and also ab < 0.

Equation (6) upon completing squares results in

$$F(x,y,z) = \frac{-(x + \frac{p}{a})^2}{-1/a} + \frac{(y + \frac{q}{b})^2}{\frac{1}{b}} + \frac{(z + \frac{r}{b})^2}{\frac{1}{b}} = 1$$

where $d = \frac{p^2}{a} + \frac{q^2}{b} + \frac{r^2}{b} - 1$

If a < 0, i.e, b > 0, intersection of the hyperboloid with plane 1, i.e., x = k, where $-p/a - \sqrt{\frac{-1}{a}} < k < -p/a + \sqrt{\frac{-1}{a}}$, would result in

$$\frac{(y + \frac{q}{b})^2}{\frac{1}{b}} + \frac{(z + \frac{r}{b})^2}{\frac{1}{b}} = 1 + \frac{(k + \frac{p}{a})^2}{\frac{-1}{a}} \tag{7}$$

where -1/a is a positive quantity. Equation (7) is that of a circle.

Intersection of the hyperboloid with plane 2, i.e., y = k, where $-q/b - \sqrt{\frac{-1}{b}} < k <$

$-q/b + \sqrt{\frac{-1}{b}}$, would generate

$$\frac{(x + \frac{p}{a})^2}{\frac{-1}{a}} - \frac{(z + \frac{r}{b})^2}{\frac{1}{b}} = \frac{(k + \frac{q}{b})^2}{\frac{-1}{b}} - 1$$

where -1/a again is a positive quantity. This equation is that of a hyperbola.

Unlike the hyperboloid of one sheet, the hyperboloid of two sheets consists of two separate pieces. Equation of a hyperboloid of two sheets lying on a plane parallel to the yz plane is

$$ax^2 + by^2 + bz^2 + 2px + 2qy + 2rz + d = 0 \tag{8}$$

where we assumed the base to be a circular one and also $ab < 0$.

As in the case of the hyperboloid of one sheet upon completing squares, equation (8) results in

$$F(x,y,z) = \frac{-(x + \frac{p}{a})^2}{-1/a} + \frac{(y + \frac{q}{b})^2}{\frac{1}{b}} + \frac{(z + \frac{r}{b})^2}{\frac{1}{b}} = -1$$

where $d = \frac{p^2}{a} + \frac{q^2}{b} + \frac{r^2}{b} + 1$. Intersection of the object with the plane 1, i.e, $x = k$, where $|k| > \sqrt{-1/a}$, would result in

$$\frac{(y + \frac{q}{b})^2}{\frac{1}{b}} + \frac{(z + \frac{r}{b})^2}{\frac{1}{b}} = -1 + \frac{(k + \frac{p}{a})^2}{\frac{-1}{a}}$$

where again -1/a is a positive quantity. This equation is that of a circle.

However when $|k| = \sqrt{(-1/a)}$, the intersection will result in a point.

Consider the case when the object is intersected with the plane 2, i.e., $y = k$, where $-q/b - \sqrt{\frac{1}{b}} < k < -q/b + \sqrt{\frac{1}{b}}$. As in the case with the hyperboloid of one sheet, the intersection in this case results in

$$\frac{(x + \frac{p}{a})^2}{\frac{-1}{a}} - \frac{(z + \frac{r}{b})^2}{\frac{1}{b}} = \frac{(k + \frac{q}{b})^2}{\frac{-1}{b}} + 1$$

which is an equation of a hyperbola.

Table 1 summarizes the various curves (conics) evolved after intersecting each of the eleven surfaces with the two planes.

TABLE 1

OBJECT	PLANE 1 : x = k	PLANE 2 : y = k
Ellipsoid	ellipse	circle
Circular cylinder	circle	line
Quadric cone	circle	hyperbola
Sphere	circle	circle
Parabolic cylinder	line	parabola
Hyperbolic cylinder	hyperbola	line
Elliptic paraboloid	ellipse	parabola
Hyperbolic paraboloid	hyperbola	parabola,line
Hyperboloid of one sheet	circle	hyperbola
Hyperboloid of two sheets	circle, point	hyperbola
Parallelepiped	line	line

As seen from table 1 above, the quadric cone, the hyperboloid of one and two sheets all generate similar curves. Also the hyperbolic cylinder and the hyperbolic paraboloid generate the same curves after being intersected with planes 1 and 2. A detailed analysis whereby the objects will be intersected with planes at various other orientation will solve the problem of objects sharing the same intersected curves.

The surface characterization described in the previous few sections have been assuming that the objects are resting in a desired stable position, however in reality objects are disoriented in space. In the next few sections we put forward our new scheme which determines the orientation of the object in space and subsequently performs the desired rotations so as to place the objects in a stable position.

DETERMINATION OF THE ROTATION MATRIX

The determination of the location and orientation of a three-dimensional object is one of the central problems in computer vision applications. It is observed that most of the methods and techniques which look into this problem require considerable preprocessing such as detecting edges or junctions, fitting curves or surfaces to segmented images and computing high order features from the input images. Most of the methods are applicable to those objects which can be described by polyhedrons. The method we propose to investigate is based on analytical geometry whereby all the rotation parameters of any object placed in any orientation in space are evaluated systematically.

Let (x,y,z) describe the coordinates of any point in our coordinate system. Consider a rotation of α about z axis, then the old coordinates in terms of the new one are represented as

$$x = x'\cos\alpha + y'\sin\alpha \tag{9}$$

$$y = -x'\sin\alpha + y'\cos\alpha \tag{10}$$

i.e., the rotation matrix is

$$R_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{11}$$

Next consider a rotation about the x′ axis by an angle β of the same point. The resultant coordinates and the old coordinates are now related by the following equations

$$z = z'\cos\beta - y''\sin\beta \tag{12}$$

$$y' = z'\sin\beta + y''\cos\beta \tag{13}$$

whereby the rotation matrix is

$$R_\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{bmatrix}. \tag{14}$$

And finally consider a rotation about the y″ axis by an angle γ, then

$$z' = z''\cos\gamma - x''\sin\gamma \tag{15}$$

$$x' = z''\sin\gamma - x''\cos\gamma \tag{16}$$

The rotation matrix for the above process was

$$R_\gamma = \begin{bmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{bmatrix}$$

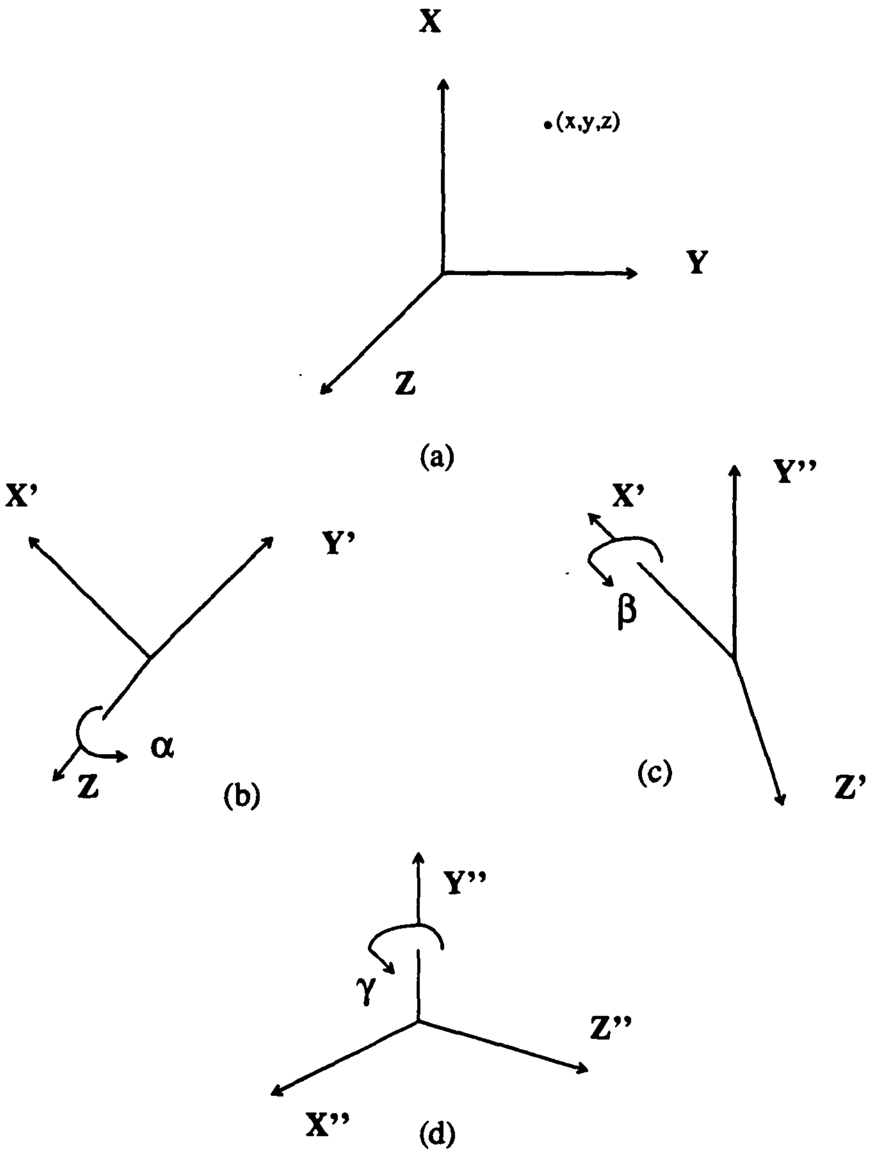
Figure 2 illustrates the behavior of a particular point (x,y,z) with respect the various rotations described above. Using equations 11, 12, 13, 14, 15, and 16 solving for x, y, and z in terms of x″, y″, and z″, yields

$$x = x''(\cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma) + y''\sin\alpha\cos\beta + z''(\sin\gamma\cos\alpha + \cos\alpha\sin\alpha\sin\beta)$$

$$y = x''(-\cos\gamma\sin\alpha - \sin\alpha\sin\beta\cos\alpha) + y''\cos\beta\cos\alpha + z''(\cos\gamma\sin\beta\cos\alpha - \sin\gamma\sin\alpha)$$

$$z = -x''\sin\gamma\cos\beta - y'\sin\beta + z'\cos\gamma\cos\beta$$

After substituting these new x, y, and z coordinates in the original quadric representa-



Figures 2(a), (b), (c), and (d) refer to the coordinate system initially, after the first rotation, the second rotation , and subsequently the third rotation respectively.

tion, we get an entire set of new coefficients for x^2, y^2, z^2, yz, xz, xy, x, y, and z.

Our immediate aim is to eliminate the rotation terms xy, yz, and xz, namely F, G, and H which have been evaluated above. Let us initially list out the new coefficients for the product terms yz, xz and xy.

$$2F' = \left[(b\cos^2\alpha + a\sin^2\alpha + h\sin 2\alpha - c)\sin 2\beta + (2g\sin\alpha + 2f\cos\alpha)\cos 2\beta\right]\cos\gamma \tag{17}$$
$$+ \left[((a - b)\sin 2\alpha + 2h\cos 2\alpha)\cos\beta - (2g\cos\alpha - 2f\sin\alpha)\sin\beta\right]\sin\gamma$$

$$2G' = \Big[a(\cos^2\alpha - \sin^2\alpha\sin^2\beta) + b(\sin^2\alpha - \cos^2\alpha\sin^2\beta) - c\cos^2\beta \tag{18}$$
$$- (g\sin\alpha\sin 2\beta + f\cos\alpha\sin 2\beta) - h\sin 2\alpha(1 + \sin^2\beta)$$
$$+ (a - b)\sin 2\alpha\sin\beta + sh\sin\beta\cos 2\alpha + \cos\beta(2g\cos\alpha - 2f\sin\alpha)\Big]\cos 2\gamma$$

$$2H' = \left[((a - b)\sin 2\alpha + 2h\cos 2\alpha)\cos\beta - (2g\cos\alpha - 2f\sin\alpha)\sin\beta\right]\cos\gamma \tag{19}$$
$$- \left[(b\cos^2\alpha + a\sin^2\alpha + h\sin 2\alpha - c)\sin 2\beta + (2g\sin\alpha + 2f\cos\alpha)\cos 2\beta\right]\sin\gamma$$

Since our aim is to eliminate the rotation terms xy, yz, and xz, let's now exclusively consider the coefficients of these rotation terms, namely F, G, and H which were evaluated above. In an iterative procedure we will be able to eliminate all of the product terms. For e.g., let's consider we wish to eliminate the term xy. The, by a specific rotation of α about the z axis, we will be able to accomplish our goal. However while carrying out this process, the orientation of the object about the two planes yz and zx, i.e., the angles the object made with these two planes have been disturbed. Now if we wish to eliminate the yz term, the object has to be rotated about the x axis by an angle beta. However in this instance, while carrying out the process, the already missing xy term reappears again, although the magnitude of its present orientation has been reduced. Hence by carrying out the above process in an iterative fashion, there comes an instance when all the coefficients of the product terms converge to zero.

Let us consider the equations (17), (18), and (19) respectively. Let us eliminate the coefficient h, i.e, the xy term in step 1. This can be accomplished by rotating the object about the z axis by an angle α, whereas $\beta=\gamma=0$. Under these circumstances the new coefficients look like as shown below

$$2f_{11} = 2g\sin\alpha_1 + 2f\cos\alpha_1 \tag{20}$$

$$2g_{11} = 2g\cos\alpha_1 - 2f\sin\alpha_1 \tag{21}$$

$$2h_{11} = (a-b)\sin 2\alpha_1 + 2h\cos 2\alpha_1 = 0 \quad \text{when} \quad \cot 2\alpha_1 = b - \frac{a}{2h} \tag{22}$$

As seen above the coefficient "h" has been forced to 0. The most significant bit of the subscript refers to the iteration number , whereas the least significant bit of the subscript reflects the number of times the object has been rotated by a specific angle. In the above case the LSB of 1 refers to the first instance where the object has been rotated by an angle α. The remaining coefficients a, b, c, p, q, and r also reflect changes brought about by the above rotation.

The new quadric equation now has a look as shown below:

$$F(x,y,z) = a_{11}x^2 + b_{11}y^2 + c_{11}z^2 + 2f_{11}yz + 2p_{11}x + 2q_{11}y + 2r_{11}z + d = 0 \tag{23}$$

Consider the second step wherein the coefficient corresponding to the yz term is forced to zero. In this particular case, the object has to be rotated by an angle β about the x axis, whereas $\alpha=\gamma=0$. Under these circumstances, the new rotation coefficients (signifying the product terms) becomes

$$2f_{12} = (b_{12} - c_{12})\sin 2\beta_1 + 2f_{11}\cos 2\beta_1 = 0 \;\; \text{if} \;\; \cot 2\beta_1 = \frac{c_{11} - b_{11}}{2}f_{11} \tag{24}$$

$$2g_{12} = 2g_{11}\cos\beta_1 \tag{25}$$

$$2h_{12} = -2g_{11}\sin\beta_1 \tag{26}$$

The new quadric equation as before looks like as shown below:

$$F(x,y,z) = a_{12}x^2 + b_{12}y^2 + c_{12}z^2 + 2f_{12}yz + 2p_{12}x + 2q_{12}y + 2r_{12}z + d = 0 \tag{27}$$

In the final step of the initial iteration, the coefficient corresponding to the xz term is forced to zero. In this case, the object is to be rotated by an angle γ about the y-axis, whereas α=β=0. Under these circumstances, the new rotation coefficients become

$$2f_{13} = 2h_{12}\sin\gamma_1 = -2g_{11}\sin\beta_1\sin\gamma_1 \tag{28}$$

$$2g_{13} =(a_{13} - c_{13)}\sin2\gamma_1 + (2g_{11}\cos\alpha_1 -2f_{11}\sin\alpha_{1)}\cos\beta_1\cos2\gamma_1 = 0 \tag{29}$$

$$\text{if} \quad \cot2\gamma_1 = \frac{c_{12} - a_{12}}{2}g_{12}$$

$$2h_{13} = 2h_{12}\cos\gamma l = -2g_{11}\sin\beta_1\cos\gamma l \tag{30}$$

Let's now carefully analyze the coefficients of xy, yz, and zx obtained in the final step of the first iteration. Consider for instance the coefficient corresponding to the yz term. It is observed that while proceeding from one step to the other, the new coefficients are getting multiplied by the sine or cosine of the concerned angle. This implies that in every succeeding steps, these coefficients are decreasing in their magnitude. To justify the above statement, let us now consider all the coefficients obtained in the second iteration.

At the end of stage 1 of the second iteration, the rotation coefficients become

$$2f_{21} = 2f_{13}\cos\alpha_2 = -2g_{11}\sin\beta_1\sin\gamma_1\cos\alpha_2 \tag{31}$$

$$2g_{21} = -2f_{13}\sin\alpha_2 = 2g_{11}\sin\beta 1\sin\gamma_1\sin\alpha_2 \tag{32}$$

$$2h_{21} = 0 \quad \text{if} \quad \cot2\alpha_2 = \frac{b_{13} - a_{13}}{2}h_{13} \tag{33}$$

At the end of the second stage of the second iteration, the rotation coefficients become

$$2f_{22} = 0 \quad \text{if} \quad \cot 2\beta_2 = \frac{c_{21} - b_{21}}{2} f_{21} \tag{34}$$

$$2g_{22} = 2g_{11}\sin\beta_1\sin\gamma_1\sin\alpha_2\cos\beta_2 \tag{35}$$

$$2h_{22} = -2g_{11}\sin\beta_1\sin\gamma_1\sin\alpha_2\sin\beta_2 \tag{36}$$

At the end of the final stage of the second iteration, the rotation coefficients reduce to

$$2f_{23} = -2g_{11}\sin\beta_1\sin\gamma_1\sin\alpha_2\sin\beta_2\sin\gamma_2 \tag{37}$$

$$2g_{23} = 0 \quad \text{if} \quad \cot 2\alpha_2 = \frac{b_{13} - a_{13}}{2} h_{13} \tag{38}$$

$$2h_{23} = -2g_{11}\sin\beta_1\sin\gamma_1\sin\alpha_2\sin\beta_2\cos\gamma_2 \tag{39}$$

where α_2, β_2, and γ_2 are the respective rotation angles along the z, x and y axes in the second iteration. Hence it is observed that with each iteration the rotation coefficients get smaller and smaller in magnitude and eventually drop out.

We are now in a position to formulate a rotation matrix whose elements correspond to the direction cosines of the x, y, and z axes of the rotated object.

$$\text{Rotation Matrix} = R_\gamma R_\beta R_\alpha \tag{40}$$

where

$$R_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{41}$$

$$R_\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{bmatrix} \tag{42}$$

$$R_\gamma = \begin{bmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{bmatrix} \tag{43}$$

Subsequently,

$$R_\alpha R_\beta R_\alpha = \begin{bmatrix} \cos\alpha\cos\gamma + \sin\alpha\sin\beta\sin\gamma & \cos\gamma\sin\alpha - \sin\gamma\sin\beta\cos\alpha & \sin\gamma\cos\beta \\ -\cos\beta\sin\alpha & \cos\beta\cos\alpha & \cos\beta \\ -\sin\gamma\cos\alpha + \cos\gamma\sin\alpha\sin\beta & -\sin\alpha\sin\gamma - \cos\gamma\sin\beta\cos\alpha & \cos\beta\cos\gamma \end{bmatrix}$$

where

$\alpha = \sum_{i=1}^{n}\alpha_i$, $\beta = \sum_{i=1}^{n}\beta_i$, and $\gamma = \sum_{i=1}^{n}\gamma_i$. n corresponds to the iteration where all the rotation terms go to zero.

Once the rotation terms, i.e., xy, yz, and xz are eliminated, the 3-D surface has the representation of

$$F(x,y,z) = Ax^2 + By^2 + Cz^2 + 2Px + 2Qy + 2Rz + D = 0 \tag{44}$$

where A, B, C, P, Q, and R are the coefficients evolved after the elimination of the rotation terms. As seen from above each of the product terms in the succeeding terms are getting multiplied by the sine or cosine of a concerned angle. This implies that in every succeeding steps, these coefficients are decreasing in their magnitude. Hence it remains to be seen that at which point, i.e, after how many iterations all the rotation coefficients converge to zero.

We are aware that the intersection of a solid with a 2-D plane generates a curve. What type of curve it intersects, depends on the surface and the orientation of the plane of intersection. Since we have no idea about the type of surface, the only thing we can do is to experiment intersecting the surface with a series of planes. Does that mean we have to intersect the object with planes oriented at 1° apart from each other? This will mean we have to intersect the object 360 different times.

Our most important assignment will be to derive a consistent method for determining the minimum number of planes that are necessary to intersect a given 3-D surface so that the generated conics uniquely characterizes the surface. This includes the derivation and formulation of the angular bounds for which a particular plane intersecting a surface results into the same 2-D curve.

Consider the equation of an ellipsoid (spheroid) resting on a plane parallel to yz plane and its axis of revolution parallel to y-axis :

The general representation reduces to the form

$$F(x,y,z) = ax^2 + by^2 + az^2 + 2px + 2qy + 2rz + d = 0. \tag{46}$$

which further reduces to

$$\frac{\left[x + \frac{p}{a}\right]^2}{\frac{1}{a}} + \frac{\left[y + \frac{q}{b}\right]^2}{\frac{1}{b}} + \frac{\left[z + \frac{r}{a}\right]^2}{\frac{1}{a}} - 1 = 0 \tag{47}$$

only if $d = \frac{p^2}{a} + \frac{q^2}{b} + \frac{r^2}{a} - 1$ and also $a > 0$, $b > 0$.

It should be noted that the coefficients a, b, p, q, r, and d are all known, $\sqrt{\frac{1}{a}}$ and $\sqrt{\frac{1}{b}}$ are the semi-major and minor axes of the ellipsoid, and [-p/a, -q/b, -r/a] are the coordinates of the center of the ellipsoid.

Consider the intersection of the ellipsoid with plane 1, i.e., x = k, where $\frac{-p}{a} - \sqrt{\frac{1}{a}} < k < \frac{-p}{a} + \sqrt{\frac{1}{a}}$,

then,

$$\frac{(y + \frac{q}{b})^2}{\frac{1}{b} - \frac{(ak+p)^2}{ab}} + \frac{(z + \frac{r}{a})^2}{\frac{1}{a} - \left[\frac{ak+p}{a}\right]^2} - 1 = 0 \tag{48}$$

which is the equation of an ellipse.

Let's now consider the intersection of the ellipsoid with plane 2, i.e., y=k, where

$$\frac{-q}{b} - \sqrt{\frac{1}{b}} < k < \frac{-q}{b} + \sqrt{\frac{1}{b}},$$

then,

$$\frac{(x + \frac{p}{a})^2}{\frac{1}{a} - \frac{(bk + q)^2}{ab}} + \frac{(z + \frac{r}{a})^2}{\frac{1}{a} - \frac{(bk + q)^2}{ab}} - 1 = 0 \qquad (49)$$

which is the equation of a circle.

To what extent the above mentioned two planes are able to distinguish the various surfaces is to be investigated.

Since different quadrics may generate similar curves, a detailed analysis whereby the objects will be intersected with planes at various other orientation will solve the problem of objects sharing the same intersected curves. Regions wherein a particular 3-D surface yields a particular type of curve is to be observed and finally we hope to represent all the 3-D surfaces mentioned earlier by a unique five tuple vector.

3–D DISCRIMINANT

In the following few sections we will investigating in detail a 3-D approach of classification and reduction of quadrics [5], which looks into the various invariants of the quadratic form under translation and rotation of 3-D objects.

The general equation of the second degree in three variables x, y, and z can be written in the form

$$F(x,y,z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2px + 2qy + 2rz + d = 0$$

Associated with f(x,y,z) are two matrices "e" and "E", such as,

$$e = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

and

$$E = \begin{bmatrix} a & h & g & p \\ h & b & f & q \\ g & f & c & r \\ p & q & r & d \end{bmatrix}$$

Let the determinant of "E" be denoted by Δ, and the determinant of "e" be denoted by "D". Also let the cofactors of each element of Δ be denoted by the corresponding capital letters. Based upon the fact that "E" is singular or non-singular, the 3-D surfaces can be singular or non-singular. Example of singular surfaces are ellipsoids, hyperboloids, paraboloids. The other quadrics are singular.

It has been shown in [5] that I, J, D, and Δ, where $I = a + b + c$, $J = \alpha + \beta + \gamma$, D, and Δ the determinants, are invariant for any general coordinate transformation. Combining these four invariants a further set of three absolute invariants, $I \frac{J}{D}$, $\frac{I^2}{J}$, and $\frac{I\,D}{\Delta}$ is also obtained.

Based upon the above given set of invariants the following classification as summarized in table 2 is obtained.

TABLE 2

Number	Surface	Rank ρ_3	Sign of Δ ρ_4	K's same sign
1	Real ellipsoid	3	4	-
2	Hyperboloid of one sheet	3	4	+
3	Hyperboloid of two sheets	3	4	-
4	Real quadric cone	3	3	
5	Elliptic paraboloid	2	4	-
6	Hyperbolic paraboloid	2	4	+
7	Real elliptic cylinder	2	3	
8	Hyperbolic cylinder	2	3	
9	Parabolic cylinder	1	3	

Results obtained using the above procedure for simulated and real objects will be discussed in the next section.

EXPERIMENTAL RESULTS AND CONCLUSIONS

For experimental purposes, range images of a sphere and a cylinder are considered. Four types of data, namely, raw image of sphere, averaged image of sphere, raw image of cylinder and averaged image of cylinder were available to us. The raw data being noisy, had to be cleared using median filters. Mask sizes of 3 x 3 and 5 x 5 were used to remove the noises and the effect of these filters on the range images were studied. Figure 3 shows the raw range image of the sphere. Figure 4 shows the raw image of a cylinder. Figures 5 and 6 are the segmented range images of the sphere and cylinder respectively. Effect of the 3 x 3 and 5 x 5 median filters on the segmented range images of the sphere and the cylinder can be seen in figures 7, 8, 9, and 10 respectively.

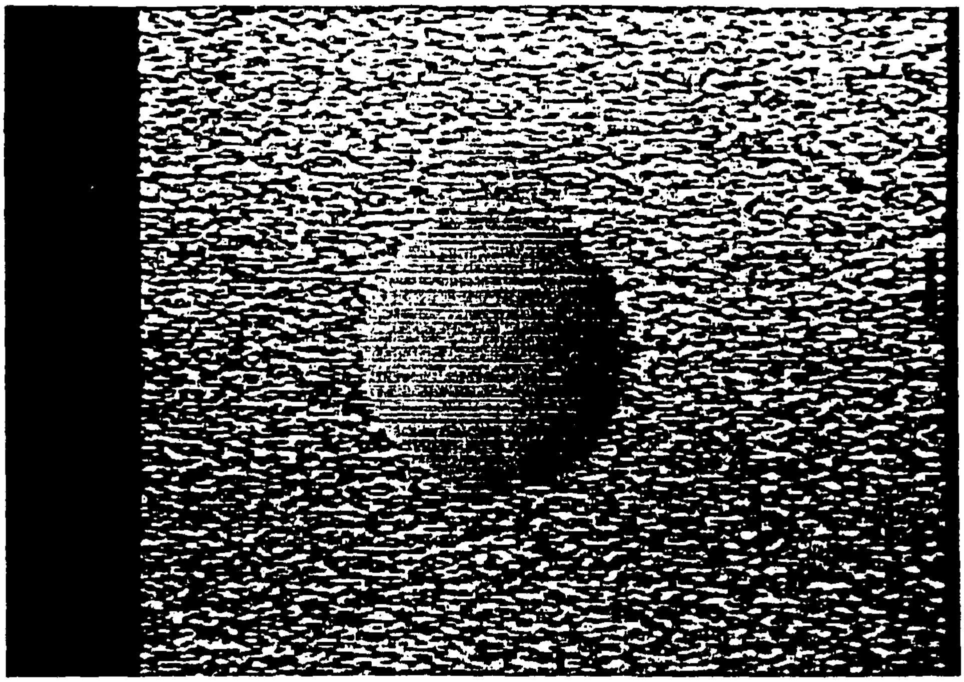


Figure 3. Raw range image of the sphere.

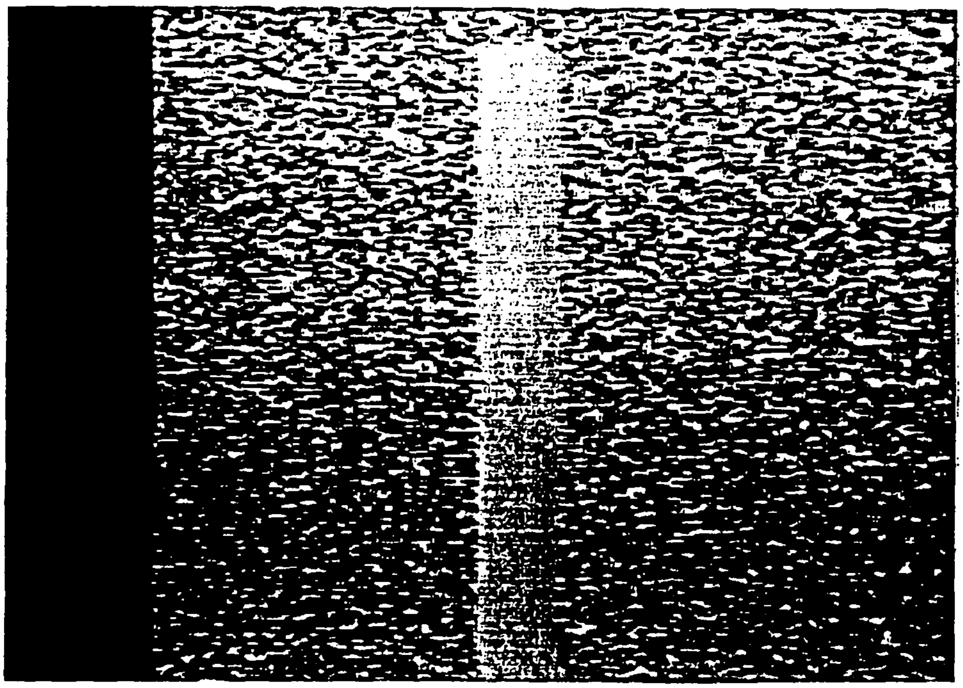


Figure 4. Raw range image of the cylinder.

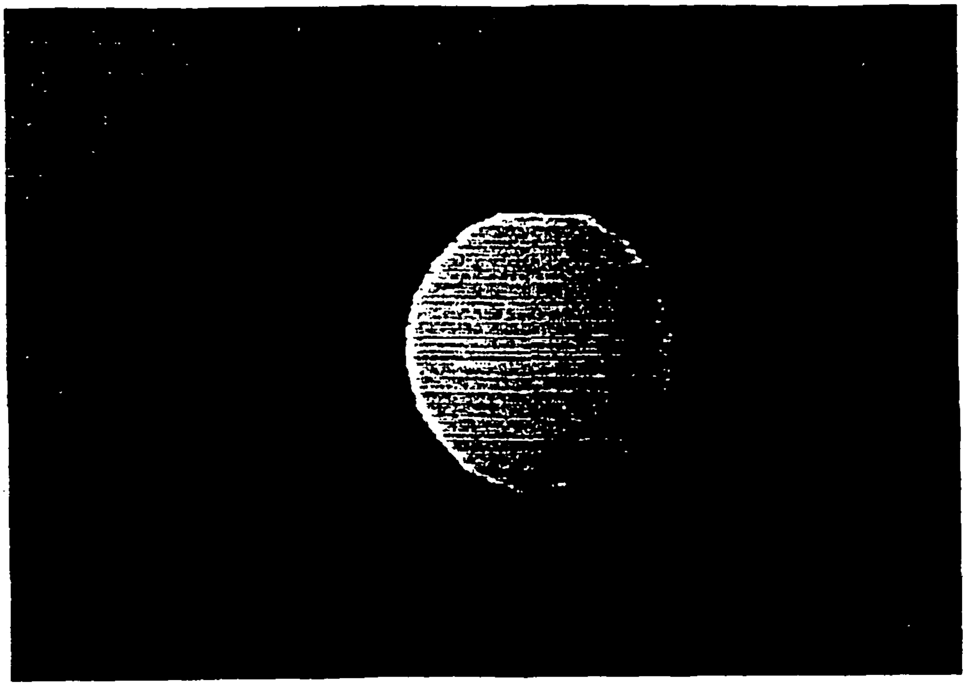


Figure 5. Segmented raw range image of the sphere.

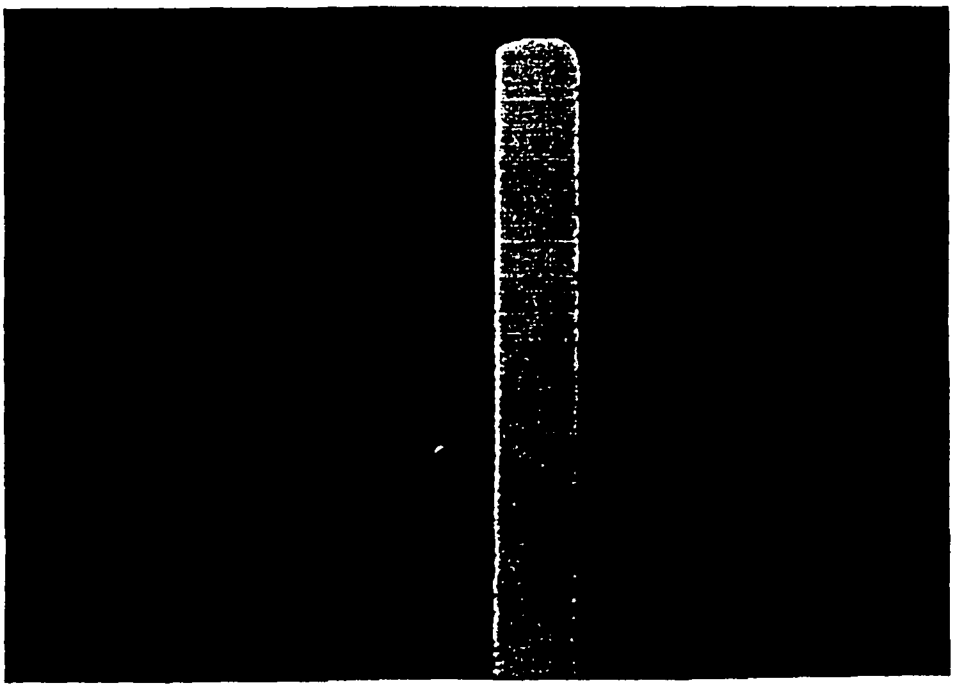


Figure 6. Segmented raw range image of the cylinder.

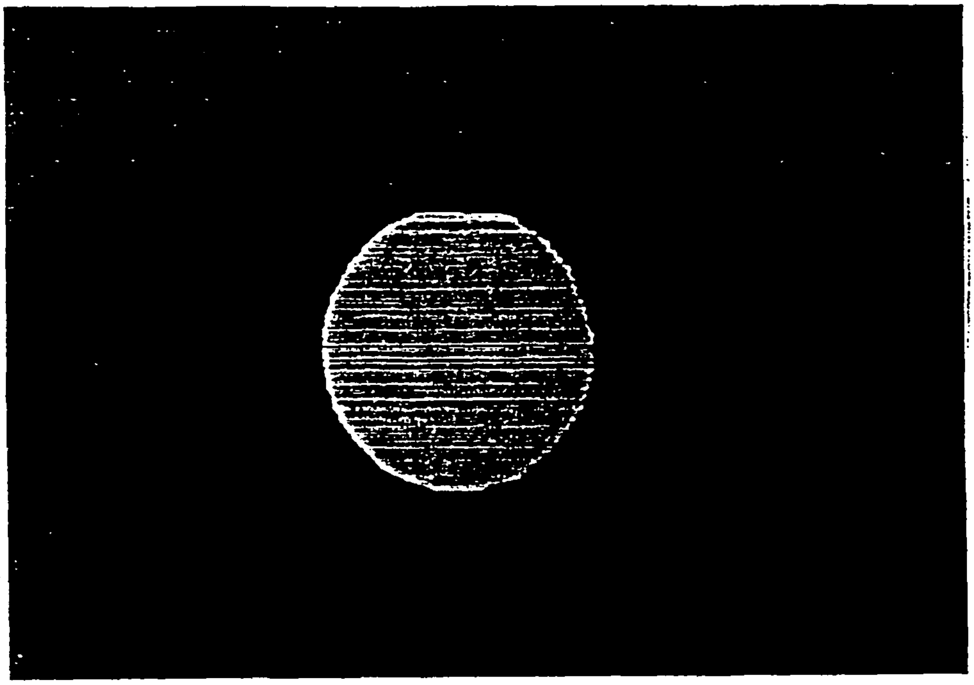


Figure 7. 3 X 3 median filtered image of the raw sphere.

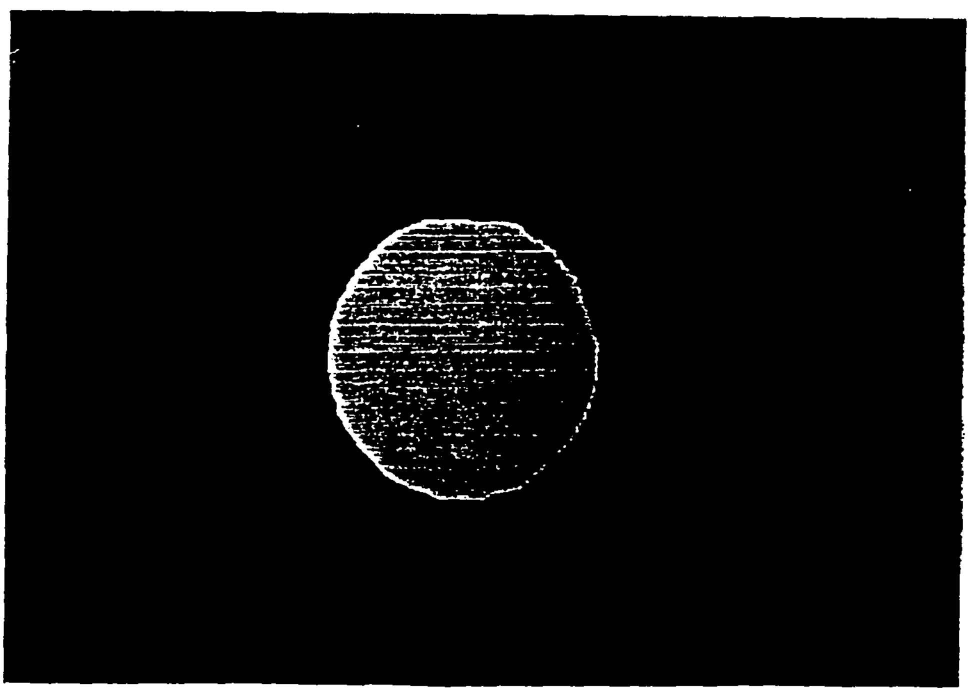


Figure 8. 5 X 5 median filtered image of the raw sphere.

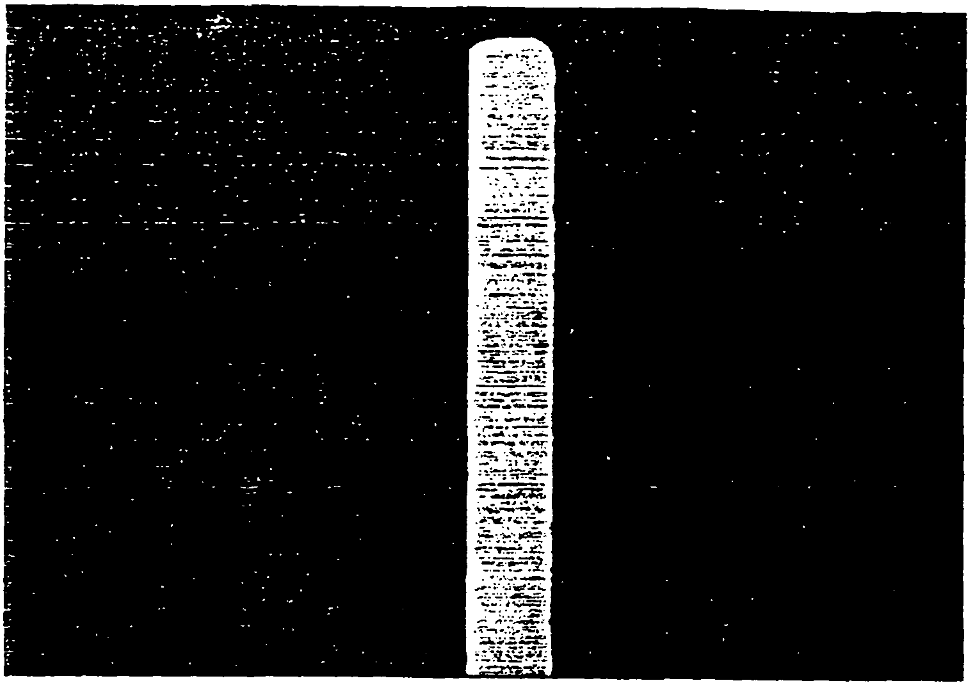


Figure 9. 3 X 3 median filtered image of the raw cylinder.

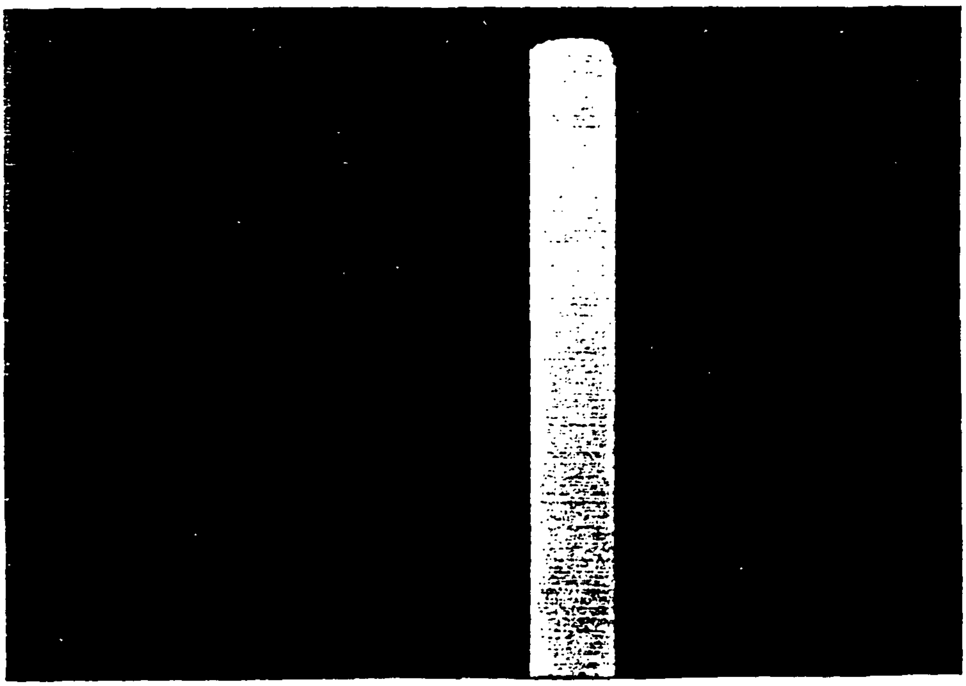


Figure 10. 5 X 5 median filtered image of the raw cylinder.

Curvature sign maps to check the integrity of the original as well as the processed images are pursued next. Not much changes were observed in the processed images. Figures 11, 12, 13, and 14 are the first and second derivatives w.r.t x and y axis for the image in figure 3.

In all of these figures, the sign "+" is assigned to a particular pixel position if the magnitude of the derivative (first or second) of that pixel is greater than the magnitude of the derivative (first or second) of the pixel to its right. Similarly the sign "-" is assigned to a particular pixel position if the magnitude of the derivative (first or second) of that pixel is less than the magnitude of the derivative (first or second) of the pixel to its right. In the case when the magnitudes of the derivatives (first or second) of either pixels is the same, the sign " " (blank) is assigned.
Two sets of coefficients were generated for each set of the range data. While obtaining the best fit plot for the raw sphere, numeral 1 refers to the situation when the first set of coefficients of the raw image fits best, numeral 2 refers to the case when the second set of coefficients of the raw image fits best, numeral 3 refers to the case when the first set of coefficients of the 3 x 3 filtered image fits best, numeral 4 occurs when the second set of coefficients of the 3 x 3 median filtered image fits best, numeral 5 occurs when the first set of coefficients for the 5 x 5 median filtered images best fits the data, and finally numeral 6 occurs in the plot when the second set of coefficients of the 5 x 5 median filtered image fits best to the data.
Tables 3 and 4 tabulate the coefficients obtained for the various images of the raw range sphere. Figure 15 shows the best fit plot for these sets of coefficients. A similar procedure is carried over for the averaged range data of the sphere.

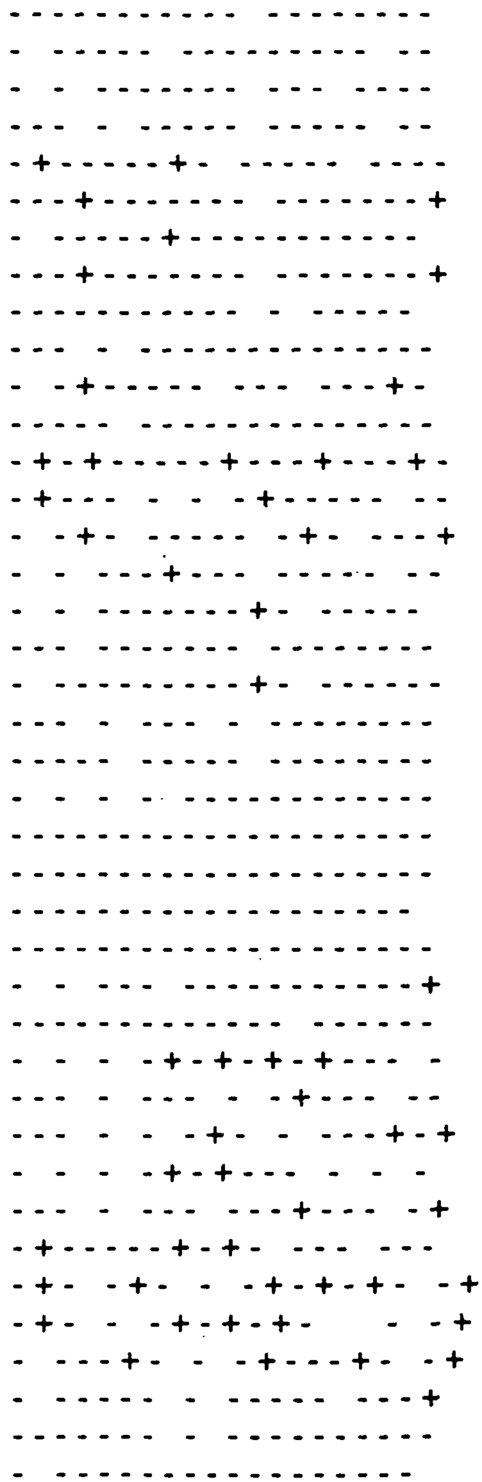


Figure 11. First derivative with respect to x-axis for the sphere raw image.

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Figure 12. First derivative with respect to the y-axis for the sphere raw image.

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Figure 13. Second derivative with respect to the x-axis for the sphere raw image.

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Figure 14. Second derivative with respect to the y-axis for the sphere raw image.

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```

Figure 15. Best fit plot for the sphere raw image.

TABLE 3

COMPARISON OF COEFFICIENTS EVALUATED FOR THE ORIGINAL AND THE PROCESSED IMAGES, SET 1			
COEFFICIENT	RAW IMAGE	3 x 3 FILTERED IMAGE	5 x 5 FILTERED IMAGE
A, COEFF. OF x^2	0.01393	0.1997	0.1409
B, COEFF. OF y^2	-0.10112	0.2455	-0.1392
C, COEFF. OF Z^2	0.5892	0.7712	0.6497
F, COEFF. OF YZ	0.8841	-0.5509	-0.3292
G, COEFF. OF XZ	-0.3664	-0.4908	-0.9748
H, COEFF. OF XY	-0.6073	0.2561	0.1359
P, COEFF. OF X	0.1136	0.1857	0.7426
Q, COEFF. OF Y	-0.4968	0.4079	0.1561
R, COEFF. OF Z	-0.5254	-0.9398	-0.8208
D. CONSTANT	0.1112	0.3130	0.2293

TABLE 4

COMPARISON OF COEFFICIENTS EVALUATED FOR THE ORIGINAL AND THE PROCESSED IMAGES, SET 2			
COEFFICIENT	RAW IMAGE	3 x 3 FILTERED IMAGE	5 x 5 FILTERED IMAGE
A, COEFF. OF X^2	0.3108	0.2027	0.3030
B, COEFF. OF y^2	0.1298	0.1475	0.0392
C, COEFF. OF Z^2	0.6218	0.7869	0.6526
F, COEFF. OF YZ	-0.4998	-0.3950	-0.4487
G, COEFF. OF XZ	-0.8032	-0.6658	-0.8376
H, COEFF. OF XY	0.3236	0.1909	0.2416
P, COEFF. OF X	0.4048	0.3344	0.4047
Q, COEFF. OF Y	0.2883	0.2665	0.2140
R, COEFF. OF Z	-0.6945	-0.9176	-0.7089
D. CONSTANT	0.1990	0.2768	0.1913

In the case of the cylinders, as with the sphere, curvature analysis, the best fit analysis etc., are carried over to see which set of coefficients can be used for the recognition process. Coefficients generated for the various images of the cylinders can be seen in appendix A.

The best fit plot for the raw sphere image prompts us to use the 3 X 3 filtered image of the second set of coefficients for the recognition process. Shown in tables 5 and 6 are the various curves intercepted with the two planes for each of these images. Each of the objects, the raw and the processed images of the sphere and the averaged and the processed images of this sphere are intersected with two planes (one parallel to the yz-plane, the other parallel to the xz-plane). A decision on the curve being an ellipse or a circle was made based upon the parity and the disparity of the x^2, y^2, and the z^2 coefficients.

TABLE 5

Sphere Images	Raw	3 x 3 filt.	5 x 5 filt.
plane 1, x = k	Ellipse	Ellipse	Circle
plane 2, y = k	Ellipse	Ellipse	Circle

TABLE 6

Sphere Images	Raw	3 x 3 filt.	5 x 5 filt.
plane 1, x = k	Ellipse	Ellipse	Circle
plane 2, y = k	Ellipse	Ellipse	Circle

Similar experiments were carried over for the raw and the averaged images of the cylinder. In the case of the cylinder, since its orientation is unknown, we applied the rotation algorithm. This algorithm eliminated the product terms and then we conducted the same set of experiments as we had done before. Results for this procedure are listed in the tables 7 and 8.

TABLE 7

Cylinder Images	Raw	3 x 3 filt.	5 x 5 filt.
plane 1, x = k	Ellipse	Line or Ellipse	Line
plane 2, y = k	Ellipse	Ellipse	Ellipse

TABLE 8

Cylinder Images	Raw	3 x 3 filt.	5 x 5 filt.
plane 1, x = k	Ellipse	Line or Ellipse	Line
plane 2, y = k	Ellipse	Ellipse	Ellipse

As mentioned in the earlier sections, we had implemented the 3-D discriminant approach for several simulated as well as synthetic data. Results for the simulated data are listed in appendix B, and seem to be very effective as predicted by the theory. However, very unsatisfactory results were obtained while experimenting with real data. Appendix C contains the program listing for the various algorithms implemented.

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APPENDIX A

```
The input file was "strawmed1.cod     "
The output file is "strawmed1.coe     "
The coeff of x-squared is  -1.1279882E-02
The coeff of y-squared is   9.9362820E-02
The coeff of z-squared is   0.9533964
The coeff of   yz      is   6.1207127E-02
The coeff of   zx      is   0.3908886
The coeff of   xy      is   7.4366599E-02
The coeff of    x      is  -0.4194884
The coeff of    y      is  -8.1451185E-02
The coeff of    z      is  -1.664232
The constant    d      is   0.7509170
```

```
The input file was "strawmed2.cod     "
The output file is "strawmed2.coe     "
The coeff of x-squared is  -6.1688922E-02
The coeff of y-squared is   0.3596755
The coeff of z-squared is   0.3823600
The coeff of   yz      is  -0.7434295
The coeff of   zx      is  -0.6636371
The coeff of   xy      is   0.6694452
The coeff of    x      is   0.8502086
The coeff of    y      is   0.6296228
The coeff of    z      is  -0.6596514
The constant    d      is   0.2292093
```

```
The input file was "STRMED31.COD        "
The output file is "STRMED31.COE       "
The coeff of x-squared is  -0.5819576
The coeff of y-squared is  -2.5062649E-02
The coeff of z-squared is  -0.4078553
The coeff of   yz      is  -9.1291323E-02
The coeff of   zx      is   0.9860787
The coeff of   xy      is   8.9543425E-02
The coeff of    x      is  -0.3951663
The coeff of    y      is   4.5018956E-02
The coeff of    z      is   0.3026683
The constant    d      is  -4.4815123E-02
```

```
The input file was "STRMED32.COD          "
The output file is "STRMED32.COE          "
The coeff of x-squared is   0.1394030
The coeff of y-squared is   0.2895423
The coeff of z-squared is   0.4720904
The coeff of   yz      is  -0.6865588
The coeff of   zx      is  -0.7503323
The coeff of   xy      is   0.5597892
The coeff of    x      is   0.6822208
The coeff of    y      is   0.5422743
The coeff of    z      is  -0.6979018
The constant    d      is   0.2410266
```

```
The input file was "STRMED51.COD      "
The output file is "STRMED51.COE      "
The coeff of x-squared is   6.2892713E-02
The coeff of y-squared is   5.5489447E-02
The coeff of z-squared is   0.9393368
The coeff of   yz      is   3.8853236E-02
The coeff of   zx      is  -0.4683146
The coeff of   xy      is   1.9889891E-02
The coeff of    x      is   0.3011477
The coeff of    y      is  -4.1780550E-02
The coeff of    z      is   -1.364659
The constant    d      is   0.5040251
```

```
The input file was "STRMED52.COD      "
The output file is "STRMED52.COE      "
The coeff of x-squared is   5.7214227E-02
The coeff of y-squared is   0.5990739
The coeff of z-squared is   0.4416272
The coeff of   yz      is  -0.8076080
The coeff of   zx      is   0.4595302
The coeff of   xy      is  -0.1490175
The coeff of    x      is  -0.5915767
The coeff of    y      is    1.089848
The coeff of    z      is   -1.019888
The constant    d      is   0.6643057
```

```
The input file was "STAMED31.COD        "
The output file is "STAMED31.COE        "
The coeff of x-squared is   0.2759137
The coeff of y-squared is   2.7527343E-02
The coeff of z-squared is   0.7029013
The coeff of   yz      is   0.1449835
The coeff of   zx      is  -0.9098228
The coeff of   xy      is  -9.6383080E-02
The coeff of    x      is   0.5634921
The coeff of    y      is  -8.9731783E-02
The coeff of    z      is  -0.8506840
The constant    d      is   0.2536311
```

```
The input file was "STAMED32.COD         "
The output file is "STAMED32.COE         "
The coeff of x-squared is   0.6636790
The coeff of y-squared is   2.0903163E-02
The coeff of z-squared is  -9.2439458E-02
The coeff of   yz      is  -2.1956543E-02
The coeff of   zx      is  -0.7604508
The coeff of   xy      is   0.7227234
The coeff of    x      is   0.4242096
The coeff of    y      is  -0.2155603
The coeff of    z      is   0.3749632
The constant    d      is  -0.2534057
```

```
The input file was "STAMED51.COD        "
The output file is "STAMED51.COE        "
The coeff of x-squared is   0.1115851
The coeff of y-squared is   3.1368352E-02
The coeff of z-squared is   0.8936580
The coeff of   yz      is   0.1347357
The coeff of   zx      is  -0.5961419
The coeff of   xy      is  -4.8396215E-02
The coeff of    x      is   0.4117958
The coeff of    y      is  -9.9320240E-02
The coeff of    z      is   -1.295335
The constant    d      is   0.4731036
```

```
The input file was "STAMED52.COD      "
The output file is "STAMED52.COE      "
The coeff of x-squared is  -4.7458898E-02
The coeff of y-squared is   0.3875865
The coeff of z-squared is   0.6523174
The coeff of   yz      is  -0.7277786
The coeff of   zx      is   0.3749601
The coeff of   xy      is  -0.4168406
The coeff of    x      is  -0.5613320
The coeff of    y      is   0.9560621
The coeff of    z      is  -1.280479
The constant    d      is   0.7334089
```

```
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```

Best fit plot for the averaged sphere image.

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15111111111111111111113133333333333333331
31331111111111311133333131333333333333531
31311111111111111111333331313333333333355 31

Best fit plot for the cylinder raw image.

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Best fit plot for the averaged cylinder image.

APPENDIX B

SURFACE CHARACTERIZATION USING 3-D DISCRIMINANT APPROACH FOR SIMULATED DATA								
COEFFICIENTS OF THE SIMULATED OBJECTS								
A, COEFF. OF X^2	1	0	0	1	0	0	1	3
B, COEFF. OF y^2	4	1	0	3	0	0	0	0
C, COEFF. OF Z^2	9	20	0	2	6	1	3	2
F, COEFF. OF YZ	-6	-4.5	1.5	-1	1.5	3	1	0
G, COEFF. OF XZ	3	-2.5	1	0	1	-2	-1	-2
H, COEFF. OF XY	-2	0.5	0.5	1	0.5	1	0	0
P, COEFF. OF X	1	0.5	-1	2	-2	2	0	0
Q, COEFF. OF Y	7	0	0	5	3	3	1	1
R, COEFF. OF Z	0	0	3	0	0	0	3	0
D, CONSTANT	10	0	0	8	0	12	9	19
OBJECT IS	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)

Object

(1) refers to PARABOLIC CYLINDER

(2) refers to HYPERBOLIC PARABOLOID

(3) refers to HYPERBOLOID OF ONE SHEET

(4) refers to ELLIPSOID

(5) refers to HYPERBOLIC CYLINDER

(6) refers to REAL QUADRIC CONE

(7) refers to HYPERBOLOID OF TWO SHEETS

(8) refers to ELLIPTIC PARABOLOID

C************** SAMPLE DATA OF 3-D DISCRIMINANT PROGRAM

```
 Coeff. of x^2 (A):
A =

    1

 Coeff. of y^2 (B):
B =

    4

 Coeff. of z^2 (C):
C =

    9

 Coeff. of yz (F):
F =

   -6

 Coeff. of xz (G):
G =

    3

 Coeff. of xy (H):
H =

   -2

 Coeff. of x (P):
P =

    1

 Coeff. of y (Q):
```

Q =

7

Coeff. of z (R):
R =

0

Constant of prop. (D):
D =

10

e =

$$\begin{matrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 3 & -6 & 9 \end{matrix}$$

EE =

$$\begin{matrix} 1 & -2 & 3 & 1 \\ -2 & 4 & -6 & 7 \\ 3 & -6 & 9 & 0 \\ 1 & 7 & 0 & 10 \end{matrix}$$

dt_e =

0

dt_EE =

0

K_K =

-0.0000
0.0000
14.0000

rho_3 =

1

rho_4 =

3

s_d_EE =

0

s1 =

-1

s2 =

1

s3 =

1

flag =

0

The sign of the ch. roots are not the same

The rank of EE is : 1.0000

The rank of e is : 3.0000

The sign of the determinant of EE is : 0.0000

The characterstics roots have the same sign? : 0.0000

The object is a PARABOLIC CYLINDER

APPENDIX C

```
C*****  Implementation of the 3-D discriminant approach
diary on
input('Coeff. of x^2 (A): ');
A=ans
input('Coeff. of y^2 (B): ');
B=ans
input('Coeff. of z^2 (C): ');
C=ans
input('Coeff. of yz (F): ');
F=ans
input('Coeff. of xz (G): ');
G=ans
input('Coeff. of xy (H): ');
H=ans
input('Coeff. of x (P): ');
P=ans
input('Coeff. of y (Q): ');
Q=ans
input('Coeff. of z (R): ');
R=ans
input('Constant of prop. (D): ');
D=ans
F=F/2;
G=G/2;
H=H/2;
P=P/2;
Q=Q/2;
R=R/2;

e=[A  H  G
H  B  F
G  F  C ]

EE=[ A  H  G  P
H  B  F  Q
G  F  C  R
P  Q  R  D  ]

dt_e=det(e)
dt_EE=det(EE)
K_K=eig(e)
rho_3=rank(e)
rho_4=rank(EE)
s_d_EE=sign(dt_EE)
s1=sign(K_K(1))
s2=sign(K_K(2))
```

```
s3=sign(K_K(3))
flag=0
if s1 == s2;
flag=flag+1
end;
if s1 == s3;
flag=flag+1
end;
if flag == 2;
an_w=1;
fprintf('\n\n  The sign of the ch. roots are the same \n')
else;
an_w=0;
end;
fprintf('\n\n  The sign of the ch. roots are not the same \n')



fprintf('\n\n     The rank of EE is : %9.4f \n ', rho_3 )
fprintf('\n\n     The rank of e is   : %9.4f \n ', rho_4 )
fprintf('\n\n     The sign of the determinant of EE is : %9.4f \n',s_d_EE )
fprintf('\n\n     The characterstics roots have the same sign? : %9.4f \n', an_w)

if rho_3==3
if rho_4==4
if s_d_EE==-1
if an_w==1
fprintf('\n\n  The object is an ELLIPSOID \n\n')
end
end
end
end
if rho_3==3
if rho_4==4
if s_d_EE==1
if an_w==0
fprintf('\n\n  The object is a HYPERBOLOID OF ONE SHEET \n\n')
end
end
end
end
if rho_3==3
if rho_4==4
if s_d_EE==-1
```

```
if an_w==0
fprintf('\n\n  The object is a HYPERBOLOID OF TWO SHEETS \n')
end
end
end
end
if rho_3==3
if rho_4==3
if an_w==0
fprintf('\n\n  The object is a REAL QUADRIC CONE \n')
end
end
end
if rho_3==2
if rho_4==4
if s_d_EE==-1
if an_w==0
fprintf('\n\n  The object is an ELLIPTIC PARABOLOID \n')
end
end
end
end
if rho_3==2
if rho_4==4
if s_d_EE==1
if an_w==0
fprintf('\n\n  The object is a HYPERBOLIC PARABOLOID \n')
end
end
end
end
if rho_3==2
if rho_4==3
if an_w==1
fprintf('\n\n  The object is an ELLIPTIC CYLINDER \n')
end
end
end
if rho_3==2
if rho_4==3
if an_w==0
fprintf('\n\n  The object is a HYPERBOLIC CYLINDER \n')
end
end
end
```

```
if rho_3==1
if rho_4==3
fprintf('\n\n  The object is a PARABOLIC CYLINDER \n')
end
end
diary off
```

```
c***** Implementation of the 3-D discriminant appoach
diary on
input('Coeff. of x^2 (A): ');
A=ans
input('Coeff. of y^2 (B): ');
B=ans
input('Coeff. of z^2 (C): ');
C=ans
input('Coeff. of yz (F): ');
F=ans
input('Coeff. of xz (G): ');
G=ans
input('Coeff. of xy (H): ');
H=ans
input('Coeff. of x (P): ');
P=ans
input('Coeff. of y (Q): ');
Q=ans
input('Coeff. of z (R): ');
R=ans
input('Constant of prop. (D): ');
D=ans
F=F/2;
G=G/2;
H=H/2;
P=P/2;
Q=Q/2;
R=R/2;

e=[A H G
H B F
G F C]

EE=[ A H G P
H B F Q
G F C R
P Q R D ]

dt_e=det(e)
dt_EE=det(EE)
K_K=eig(e)
rho_3=rank(e)
rho_4=rank(EE)
s_d_EE=sign(dt_EE)
s1=sign(K_K(1))
s2=sign(K_K(2))
```

```
s3=sign(K_K(3))
flag=0
if s1 == s2;
flag=flag+1
end;
if s1 == s3;
flag=flag+1
end;
if flag == 2;
an_w=1;
fprintf('\n\n  The sign of the ch. roots are the same \n')
else;
an_w=0;
end;
fprintf('\n\n  The sign of the ch. roots are not the same \n')



fprintf('\n\n    The rank of EE is : %9.4f \n ', rho_3 )
fprintf('\n\n    The rank of e is  : %9.4f \n ', rho_4 )
fprintf('\n\n    The sign of the determinant of EE is : %9.4f \n',s_d_EE )
fprintf('\n\n    The characterstics roots have the same sign? : %9.4f \n', an_w)

if rho_3==3
if rho_4==4
if s_d_EE==-1
if an_w==1
fprintf('\n\n  The object is an ELLIPSOID \n\n')
end
end
end
end
if rho_3==3
if rho_4==4
if s_d_EE==1
if an_w==0
fprintf('\n\n  The object is a HYPERBOLOID OF ONE SHEET \n\n')
end
end
end
end
if rho_3==3
if rho_4==4
if s_d_EE==-1
```

```
if an_w==0
fprintf('\n\n  The object is a HYPERBOLOID OF TWO SHEETS \n')
end
end
end
end
if rho_3==3
if rho_4==3
if an_w==0
fprintf('\n\n  The object is a REAL QUADRIC CONE \n')
end
end
end
if rho_3==2
if rho_4==4
if s_d_EE==-1
if an_w==0
fprintf('\n\n  The object is an ELLIPTIC PARABOLOID \n')
end
end
end
end
if rho_3==2
if rho_4==4
if s_d_EE==1
if an_w==0
fprintf('\n\n  The object is a HYPERBOLIC PARABOLOID \n')
end
end
end
end
if rho_3==2
if rho_4==3
if an_w==1
fprintf('\n\n  The object is an ELLIPTIC CYLINDER \n')
end
end
end
if rho_3==2
if rho_4==3
if an_w==0
fprintf('\n\n  The object is a HYPERBOLIC CYLINDER \n')
end
end
end
```

```
if rho_3==1
if rho_4==3
fprintf('\n\n  The object is a PARABOLIC CYLINDER \n')
end
end
diary off
```

```
c**** PROGRAM SURFACE ALIGNMENT
c**** This program is used to ged rid of the product terms
c**** from the quadratic representation of any 3D surface.
c**** The new coefficents generated consisits of the square terms
c**** and the x,y ,z, and the constant term.
c**** Last compiled on 14feb. 1991.

      REAL AA,BB,CC,DD,FF,GG,HH,PP,QQ,RR,D,Test_f,Test_g,test_h
      REAL A(50,50),B(50,50),C(50,50),F(50,50)
      REAL G(50,50),H(50,50),ALPHA(100),BETA(100)
      REAL RESULT(200,200),P(50,50),Q(50,50),R(50,50)
      REAL    AAA,BBB,CCC,DDD,EEE,FFF,GGG,HHH,III,ROT(3,3)
      REAL DEL1,DEL2,DEL3,A_A,B_B,C_C,F_F,G_G,H_H,GAMMA(100)
      REAL VV,VVV,VVVV,VVVVV,THRESHLD,INITMIN,ABSA,ABSB,ABSC
      REAL    A_AA,B_BB,C_CC,D_DD,P_PP,Q_QQ,R_RR
      REAL ABSF,ABSG,ABSH,ABSP,ABSQ,ABSR,RRR(50),alptot,bettot
      REAL gamtot
      INTEGER N,M,I,J
C     F(X,Y,Z)=Ax**2+By**2+Cz**2+2Fyz+2Gxz+2Hxy+2Px+2Qy+2Rz+D
C     =0
C     PARAMETER       (THRESHLD= 0.0000000000000001)

      OPEN(UNIT=1,FILE='CONVERGENCE.DAT',STATUS='NEW')
      TYPE*,'ENTER VALUE FOR THRESHLD:'
      ACCEPT*,THRESHLD

      Type*,'Enter coef. of x ** 2 :'
      Accept*,AA
      Type*,'Enter coef. of y ** 2 :'
      Accept*,BB
      Type*,'Enter coef. of z ** 2 :'
      Accept*,CC
      Type*,'Enter coef. of yz :'
      Accept*,FF
      Type*,'Enter coef. of xz :'
      Accept*,GG
      Type*,'Enter coef. of xy :'
      Accept*,HH
      Type*,'Enter coef. of x :'
      Accept*,PP
      Type*,'Enter coef. of y :'
      Accept*,QQ
      Type*,'Enter coef. of z :'
      Accept*,RR
      Type*,'Enter constant of prop. :'
      Accept*,D
```

```
      A(1,1)=AA
      B(1,1)=BB
      C(1,1)=CC
      F(1,1)=FF
      G(1,1)=GG
      H(1,1)=HH
      P(1,1)=PP
      Q(1,1)=QQ
      R(1,1)=RR
      ABSA=ABS(AA)
      ABSB=ABS(BB)
      ABSC=ABS(CC)
      ABSF=ABS(FF)
      ABSG=ABS(GG)
      ABSH=ABS(HH)
      ABSP=ABS(PP)
      ABSQ=ABS(QQ)
      ABSR=ABS(RR)
      RRR(1)=ABSA
      RRR(2)=ABSB
      RRR(3)=ABSC
      RRR(4)=ABSF
      RRR(5)=ABSG
      RRR(6)=ABSH
      RRR(7)=ABSP
      RRR(8)=ABSQ
      RRR(9)=ABSR
      DO 3980 I=1,9
      IF (RRR(I).EQ.0)THEN
      RRR(I)=10000
      ENDIF
3980  CONTINUE
      INITMIN=AMIN1(RRR(1),RRR(2),RRR(3),RRR(4),RRR(5),RRR(6),RRR(7)
     + ,RRR(8),RRR(9))
      WRITE(*,*)INITMIN
      IF (ABS(INITMIN).LT.1.0)THEN
      A(1,1)=A(1,1)/INITMIN
      B(1,1)=B(1,1)/INITMIN
      C(1,1)=C(1,1)/INITMIN
      F(1,1)=F(1,1)/INITMIN
      G(1,1)=G(1,1)/INITMIN
      H(1,1)=H(1,1)/INITMIN
      P(1,1)=P(1,1)/INITMIN
      Q(1,1)=Q(1,1)/INITMIN
```

```
      Q(1,1)=Q(1,1)/INITMIN
      DD_D=D/INITMIN
      ELSE
      GOTO 3405
      ENDIF
3405  A(1,1)=AA
      B(1,1)=BB
      C(1,1)=CC
      F(1,1)=FF
      G(1,1)=GG
      H(1,1)=HH
      P(1,1)=PP
      Q(1,1)=QQ
      R(1,1)=RR


345   if (b(1,1).eq.a(1,1)) then

      goto 1167
      else
c     goto 57
c     endif
c     else
      goto 57
      endif
57    alpha(1)=(0.5*ATAND((H(1,1)/(B(1,1)-A(1,1)))))
      A(1,1)=A(1,1)*COSD(ALPHA(1))*COSD(ALPHA(1))+B(1,1)*
     + SIND(ALPHA(1))*SIND(ALPHA(1))- H(1,1)*SIND(ALPHA(1))*
     + COSD(ALPHA(1))
      B(1,1)=B(1,1)*COSD(ALPHA(1))*COSD(ALPHA(1))+A(1,1)*
     + SIND(ALPHA(1))*SIND(ALPHA(1))+H(1,1)*SIND(ALPHA(1))*
     + COSD(ALPHA(1))
      C(1,1)=C(1,1)
      F(1,1)=G(1,1)*SIND(ALPHA(1))+F(1,1)*COSD(ALPHA(1))
      G(1,1)=G(1,1)*COSD(ALPHA(1))-F(1,1)*SIND(ALPHA(1))
      H(1,1)=0
      P(1,1)=P(1,1)*COSD(ALPHA(1))-Q(1,1)*SIND(ALPHA(1))
      Q(1,1)=Q(1,1)*COSD(ALPHA(1))+P(1,1)*SIND(ALPHA(1))

      R(1,1)=R(1,1)


      IF (ABS(F(1,1)).LT.THRESHLD)THEN
      GOTO 1005
```

```
      ELSE
      GOTO 1167
      ENDIF
1005  IF (ABS(G(1,1)).LT.THRESHLD)THEN
      GOTO 1812
      ELSE
      GOTO 1167
      ENDIF
1167  IF (C(1,1).EQ.B(1,1))THEN
      GOTO 1169
      ELSE
      GOTO 1200
      ENDIF
1200  BETA(1)=(0.5*ATAND((F(1,1)/(C(1,1)-B(1,1)))))
      A(1,2)=A(1,1)
      B(1,2)=B(1,1)*COSD(BETA(1))*COSD(BETA(1))+C(1,1)*
     + SIND(BETA(1))*SIND(BETA(1))-F(1,1)*SIND(BETA(1))*COSD(BETA(1))

      C(1,2)=C(1,1)*COSD(BETA(1))*COSD(BETA(1))+B(1,1)*
     + SIND(BETA(1))*SIND(BETA(1))+F(1,1)*SIND(BETA(1))*COSD(BETA(1))
      F(1,2)=0
      G(1,2)=G(1,1)*COSD(BETA(1))
      H(1,2)=-G(1,1)*SIND(BETA(1))
      P(1,2)=P(1,1)
      Q(1,2)=Q(1,1)*COSD(BETA(1))-R(1,1)*SIND(BETA(1))

      R(1,2)=R(1,1)*COSD(BETA(1))+Q(1,1)*SIND(BETA(1))
      IF (ABS(H(1,2)).LT.THRESHLD)THEN
      GOTO 1007
      ELSE
      GOTO 1169
      ENDIF
1007  IF (ABS(G(1,2)).LT.THRESHLD)THEN
      GOTO 1812
      ELSE
      GOTO 1169
      ENDIF

1169  IF (C(1,2).EQ.A(1,2))THEN
      GOTO 67
      ELSE
      GOTO 1235
      ENDIF
```

```
1235  GAMMA(1)=(0.5*ATAND((G(1,2)/(C(1,2)-A(1,2)))))
      A(1,3)=A(1,2)*COSD(GAMMA(1))*COSD(GAMMA(1))+C(1,2)*
    + SIND(GAMMA(1))*SIND(GAMMA(1))-G(1,2)*SIND(GAMMA(1))
    + *COSD(GAMMA(1))
      B(1,3)=B(1,2)
      C(1,3)=C(1,2)*COSD(GAMMA(1))*COSD(GAMMA(1))+A(1,2)*
    + SIND(GAMMA(1))*SIND(GAMMA(1))+G(1,2)*SIND(GAMMA(1))
    + *COSD(GAMMA(1))
      F(1,3)=H(1,2)*SIND(GAMMA(1))
      G(1,3)=0
      H(1,3)=H(1,2)*COSD(GAMMA(1))

      P(1,3)=P(1,2)*COSD(GAMMA(1))-R(1,2)*SIND(GAMMA(1))
      Q(1,3)=Q(1,2)
      R(1,3)=R(1,2)*COSD(GAMMA(1))+P(1,2)*SIND(GAMMA(1))


      IF (ABS(F(1,3)).LT.THRESHLD)THEN
      GOTO 1009
      ELSE
      GOTO 67
      ENDIF
1009  IF (ABS(H(1,3)).LT.THRESHLD)THEN
      GOTO 1812
      ELSE
      GOTO 67
      ENDIF


67    DO 10 I=2,100
C     DO 20 J=1

71    if((b(i-1,3).eq.a(i-1,3)))then
      goto 167
C      else
c     if(h(i,3).eq.0)then

c     goto 67
c     else
c     goto 67
c     endif
      else
      goto 177
      endif
```

```
177   alpha(I)=(0.5*ATAND((H(I-1,3)/(B(I-1,3)-A(I-1,3)))))
      A(I,1)=A(I-1,3)*COSD(ALPHA(I))*COSD(ALPHA(I))+(B(I-1,3))*
     + SIND(ALPHA(I))*SIND(ALPHA(I))- H(I-1,3)*SIND(ALPHA(I))*
     + COSD(ALPHA(I))
      B(I,1)=B(I-1,3)*COSD(ALPHA(I))*COSD(ALPHA(I))+A(I-1,3)*
     + SIND(ALPHA(I))*SIND(ALPHA(I))+H(I-1,3)*SIND(ALPHA(I))*
     + COSD(ALPHA(I))
      C(I,1)=C(I-1,3)
      F(I,1)=F(I-1,3)*COSD(ALPHA(I))
      G(I,1)=-F(I-1,3)*SIND(ALPHA(I))
      H(I,1)=0
      P(I,1)=P(I-1,3)*COSD(ALPHA(I))-Q(I-1,3)*SIND(ALPHA(I))
      Q(I,1)=Q(I-1,3)*COSD(ALPHA(I))+P(I-1,3)*SIND(ALPHA(I))
      R(I,1)=R(I-1,3)

      IF (ABS(F(I,1)).LT.THRESHLD)THEN
      GOTO 1011
      ELSE
      GOTO 167
      ENDIF
1011  IF (ABS(G(I,1)).LT.THRESHLD)THEN
      N=I
      GOTO 666
      ELSE
      GOTO 167
      ENDIF
167   if((c(i,1).eq.b(i,1)))then
      goto 69
      else
c     goto 59
c     endif
c     else
      goto 59
      endif
59    BETA(I)=(0.5*ATAND((F(I,1)/(C(I,1)-B(I,1)))))
      A(I,2)=A(I,1)
      B(I,2)=B(I,1)*COSD(BETA(I))*COSD(BETA(I))+C(I,1)*
     + SIND(BETA(I))*SIND(BETA(I))-F(I,1)*SIND(BETA(I))*COSD(BETA(I))

      C(I,2)=C(I,1)*COSD(BETA(I))*COSD(BETA(I))+B(I,1)*
     + SIND(BETA(I))*SIND(BETA(I))+F(I,1)*SIND(BETA(I))*COSD(BETA(I))
      F(I,2)=0
      G(I,2)=G(I,1)*COSD(BETA(I))
      H(I,2)=-G(I,1)*SIND(BETA(I))
      P(I,2)=P(I,1)
      Q(I,2)=Q(I,1)*COSD(BETA(I))-R(I,1)*SIND(BETA(I))
```

```
      R(I,2)=R(I,1)*COSD(BETA(I))+Q(I,1)*SIND(BETA(I))


      IF (ABS(G(I,2)).LT.THRESHLD)THEN
      GOTO 1013
      ELSE
      GOTO 69
      ENDIF
1013  IF (ABS(H(I,2)).LT.THRESHLD)THEN
      N=I
      GOTO 666
      ELSE
      GOTO 69
      ENDIF

c69   if(g(i,2).eq.0)then
69          if((c(i,2).eq.a(i,2)))then
      goto 10
      else
c     goto 63
c     endif
c     else
      goto 63
      endif
63    GAMMA(I)=(0.5*ATAND((G(I,2)/(C(I,2)-A(I,2)))))
      A(I,3)=A(I,2)*COSD(GAMMA(I))*COSD(GAMMA(I))+C(I,2)*
     + SIND(GAMMA(I))*SIND(GAMMA(I))-G(I,2)*SIND(GAMMA(I))
     + *COSD(GAMMA(I))
      B(I,3)=B(I,2)
      C(I,3)=C(I,2)*COSD(GAMMA(I))*COSD(GAMMA(I))+A(I,2)*
     + SIND(GAMMA(I))*SIND(GAMMA(I))+G(I,2)*SIND(GAMMA(I))
     + *COSD(GAMMA(I))
      F(I,3)=H(I,2)*SIND(GAMMA(I))
      G(I,3)=0
      H(I,3)=H(I,2)*COSD(GAMMA(I))

      P(I,3)=P(I,2)*COSD(GAMMA(I))-R(I,2)*SIND(GAMMA(I))
      Q(I,3)=Q(I,2)
      R(I,3)=R(I,2)*COSD(GAMMA(I))+P(I,2)*SIND(GAMMA(I))

      IF (ABS(F(I,3)).LT.THRESHLD)THEN
      GOTO 1015
      ELSE
      GOTO 10
      ENDIF
1015  IF (ABS(H(I,3)).LT.THRESHLD)THEN
```

```
      N=I
      GOTO 666
      ELSE
      GOTO 10
      ENDIF


20    CONTINUE
10    CONTINUE
1812  N=1
666   WRITE(*,*)N
      WRITE(*,123)
123   FORMAT(5X,'*******************************************************
   +  *************')
      WRITE(*,*)('THE NUMBER OF ITERATIONS COMPLETED IS:',N)

      M=N*3
      DO 1000 I=1,N
      DO 1001 J=1,3
      RESULT(3*(I-1)+J,1)=A(I,J)
      RESULT(3*(I-1)+J,2)=B(I,J)
      RESULT(3*(I-1)+J,3)=C(I,J)
      RESULT(3*(I-1)+J,4)=F(I,J)
      RESULT(3*(I-1)+J,5)=G(I,J)
      RESULT(3*(I-1)+J,6)=H(I,J)
      RESULT(3*(I-1)+J,7)=P(I,J)
      RESULT(3*(I-1)+J,8)=Q(I,J)
      RESULT(3*(I-1)+J,9)=R(I,J)

1001  CONTINUE
1000  CONTINUE
      WRITE(1,*)('THE NUMBER OF ITERATIONS COMPLETED IS :',N)
      WRITE(1,123)


      WRITE(1,*)('COEFF. OF X SQUARE TERM IS : ', AA)
198   WRITE(1,*)('COEFF. OF Y SQUARE TERM IS : ', BB)
298   WRITE(1,*)('COEFF. OF Z SQUARE TERM IS : ', CC)
398   WRITE(1,*)('COEFF. OF YZ SQUARE TERM IS : ', FF)
498   WRITE(1,*)('COEFF. OF XZ SQUARE TERM IS : ' ,GG)
598   WRITE(1,*)('COEFF. OF XY SQUARE TERM IS : ' ,HH)
      WRITE(1,*)('COEFF. OF X TERM IS : ', PP)
      WRITE(1,*)('COEFF. OF Y  TERM IS : ', QQ)
      WRITE(1,*)('COEFF. OF Z TERM IS : ', RR)
      WRITE(1,*)('CONSTANT OF PROP. IS : ', D)
```

```
      write(1,123)
      write(1,123)
      write(1,123)
      DO 2000 I=1,M
      WRITE(1,*)(RESULT(I,J),J=1,9)
2000  CONTINUE

      A_AA=RESULT(M-2,1)
      B_BB=RESULT(M-2,2)
      C_CC=RESULT(M-2,3)
      P_PP=RESULT(M-2,7)
      Q_QQ=RESULT(M-2,8)
      R_RR=RESULT(M-2,9)
      D_DD=D

      do 30001 i=1,3
      write(1,123)
30001 continue
      WRITE(1,*)('THE NEW COEFF. OF X SQUARE TERM IS : ', A_AA)
      WRITE(1,*)('THE NEW COEFF. OF Y SQUARE TERM IS : ', B_BB)
      WRITE(1,*)('THE NEW COEFF. OF Z SQUARE TERM IS : ', C_CC)
      WRITE(1,*)('THE NEW COEFF. OF X TERM IS : ', P_PP)
      WRITE(1,*)('THE NEW COEFF. OF Y  TERM IS : ', Q_QQ)
      WRITE(1,*)('THE NEW COEFF. OF Z TERM IS : ', R_RR)
      WRITE(1,*)('THE NEW CONSTANT OF PROP. IS : ',D_DD)

      do 3001 i=1,3
      write(1,123)
3001  continue

      write(1,1278)
1278  format(6x,'A',9x,'B',9x'C',9x,'F',9x,'G',9x,'H',9x,'P',
    + 9x,'Q',9x,'R')
      write(1,1897)
1897  format(5x,'-----------------------------------------------------
    + ---------')


      DO 2001 I=1,M
      WRITE(1,1234)(RESULT(I,J),J=1,9)
2001  CONTINUE
1234  format(9F10.5)
      DO 3000 I=1,5
      WRITE(1,123)
3000  CONTINUE
```

```
      write(1,1908)
1908  format(6x,'Alpha',9x,'Beta',9x,'Gamma')
      write(1,1897)
      DO 4000 I=1,N
      WRITE(1,*)ALPHA(I),BETA(I),GAMMA(I)
4000  CONTINUE
      alptot=alpha(1)+alpha(2)+alpha(3)
      bettot=beta(1)+beta(2)+beta(3)
      gamtot=gamma(1)+gamma(2)+gamma(3)
      write(1,123)
      write(1,123)
      write(1,1998)
1998  format(6x,'ALPTOT',9x,'BETTOT',9x,'GAMTOT')
      write(1,*)alptot,bettot,gamtot
      write(1,123)
c*****        To evaluate coeff. of yz, xz, and xy once alpha, beta
c*****  and gamma are evaluated.
      write(*,*)alpha(1),beta(1),gamma(1)
      AAA=BB*cosd(alpha(1))*cosd(alpha(1))+(AA*sind(alpha(1))
     + *sind(alpha(1)))+((HH/2)*sind(2*alpha(1)))-CC
      BBB=gg*sind(alpha(1))+(ff*cosd(alpha(1)))
      CCC=((aa-bb))*sind(2*alpha(1))+(hh*cosd(2*alpha(1)))
      DDD=gg*cosd(alpha(1))-(ff*sind(alpha(1)))
      EEE=aa*(cosd(alpha(1))*cosd(alpha(1))-(sind(alpha(1))
     + *sind(alpha(1))*sind(beta(1))*sind(beta(1))))
      FFF=bb*(sind(alpha(1))*sind(alpha(1))-(cosd(alpha(1))
     + *cosd(alpha(1))*sind(beta(1))*sind(beta(1))))
      GGG=cc*cosd(beta(1))*cosd(beta(1))
      HHH=(gg/2)*sind(alpha(1))*sind(2*beta(1))+((ff/2)*cosd(alpha(1))*
     + sind(2*beta(1)))
      III=(hh/2)*sind(2*alpha(1))*(1+sind(beta(1))*
     + sind(beta(1)))

      Test_F=(AAA*sind(2*beta(1))+BBB*cosd(2*beta(1)))*
     + cosd(gamma(1))+(CCC*cos(beta(1))-DDD*sind(beta(1)))
     + *sind(gamma(1))

      Test_G=(EEE+FFF-GGG-HHH-III)*SIND(2*GAMMA(1)) +
     + (CCC*SIND(BETA(1))+COSD(BETA(1))*DDD)*COSD(2*GAMMA(1))

      TEST_H=(CCC*COSD(BETA(1))-DDD*SIND(BETA(1)))*COSD(GAMMA(1))
-
     + (AAA*SIND(2*BETA(1))+BBB*COSD(2*BETA(1)))*SIND(GAMMA(1))

c     write(1,123)
      write(1,123)
```

```
c       write(1,124)
c
c       write(1,*)test_f,test_g,test_h
        write(1,123)
        write(1,123)
        write(1,124)
124     format(5x,'THE ROTATION MATRIX IS : ')
c****   To evaluate the rotation matrix

        rot(1,1)=cosd(alpha(1))*cosd(gamma(1))-(sind(alpha(1))*
     +  sind(beta(1))*sind(gamma(1)))
        rot(1,2)=-sind(alpha(1))*cosd(gamma(1))-(cosd(alpha(1))*
     +  sind(beta(1))*sind(gamma(1)))
        rot(1,3)=-sind(gamma(1))*cosd(beta(1))

        rot(2,1)=sind(alpha(1))*cosd(beta(1))
        rot(2,2)=cosd(beta(1))*cosd(alpha(1))
        rot(2,3)=-sind(beta(1))

        rot(3,1)=cosd(alpha(1))*sind(gamma(1))+(sind(alpha(1))*
     +  sind(beta(1))*cosd(gamma(1)))
        rot(3,2)=cosd(alpha(1))*cosd(gamma(1))*sind(beta(1))
     +  -(sind(alpha(1))*sind(gamma(1)))
        rot(3,3)=cosd(gamma(1))*cosd(beta(1))

        DO 989 I=1,3
        WRITE(1,*)(ROT(I,J),J=1,3)
989     CONTINUE
                stop
        end
```

THE COEFFICIENTS CONSIDERED ARE OF THE AVERAGED CYLINDER.
****
THE NUMBER OF ITERATIONS COMPLETED IS : 4
****
COEFF. OF X SQUARE TERM IS : 0.8338000
COEFF. OF Y SQUARE TERM IS : 4.1100001E-03
COEFF. OF Z SQUARE TERM IS : 5.9000000E-02
COEFF. OF YZ SQUARE TERM IS : -1.0300000E-03
COEFF. OF XZ SQUARE TERM IS : -0.6368000
COEFF. OF XY SQUARE TERM IS : 0.4437000
COEFF. OF X TERM IS : 1.1410000E-02
COEFF. OF Y TERM IS : -0.1894000
COEFF. OF Z TERM IS : 0.1932000
CONSTANT OF PROP. IS : -0.1341000
*** ************
*** ************
*** ************

```
 0.8893950     -4.8199981E-02  5.9000000E-02  0.1537948     -0.5803153
 0.0000000E+00 -3.4971744E-02 -0.1752182      0.1932000
 0.8893950     -8.8334545E-02  9.9134572E-02  0.0000000E+00 -0.5144598
 0.2685087     -3.4971744E-02 -0.2447266      9.0202741E-02
 0.9657465     -8.8334545E-02  2.2782966E-02  7.6404691E-02  0.0000000E+00
 0.2574087     -5.9193403E-02 -0.2447266      7.6522537E-02
 0.9812339     -0.1038219      2.2782966E-02  7.5857453E-02  9.1281543E-03
 0.0000000E+00 -8.8007204E-02 -0.2359019      7.6522537E-02
 0.9812339     -0.1143151      3.3276103E-02  0.0000000E+00  8.7976847E-03
-2.4339152E-03 -8.8007204E-02 -0.2477653      1.0851689E-02
 0.9812542     -0.1143151      3.3255693E-02  1.1293819E-05  0.0000000E+00
-2.4338888E-03 -8.7955907E-02 -0.2477653      1.1259941E-02
 0.9812556     -0.1143164      3.3255693E-02  1.1293812E-05 -1.2545008E-08
 0.0000000E+00 -8.7680638E-02 -0.2478629      1.1259941E-02
 0.9812556     -0.1143164      3.3255693E-02  0.0000000E+00 -1.2545008E-08
 4.8003978E-13 -8.7680638E-02 -0.2478633      1.1250457E-02
 0.9812556     -0.1143164      3.3255693E-02  3.1762148E-21  0.0000000E+00
 4.8003978E-13 -8.7680638E-02 -0.2478633      1.1250456E-02
 0.9812556     -0.1143164      3.3255693E-02  3.1762148E-21  6.9585087E-34
 0.0000000E+00 -8.7680638E-02 -0.2478633      1.1250456E-02
 0.0000000E+00  0.0000000E+00  0.0000000E+00  0.0000000E+00  0.0000000E+00
 0.0000000E+00  0.0000000E+00  0.0000000E+00  0.0000000E+00
 0.0000000E+00  0.0000000E+00  0.0000000E+00  0.0000000E+00  0.0000000E+00
 0.0000000E+00  0.0000000E+00  0.0000000E+00  0.0000000E+00
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THE NEW COEFF. OF X SQUARE TERM IS : 0.9812556
THE NEW COEFF. OF Y SQUARE TERM IS : -0.1143164
THE NEW COEFF. OF Z SQUARE TERM IS : 3.3255693E-02
THE NEW COEFF. OF X TERM IS : -8.7680638E-02
THE NEW COEFF. OF Y TERM IS : -0.2478633
THE NEW COEFF. OF Z TERM IS : 1.1250456E-02
THE NEW CONSTANT OF PROP. IS : -0.1341000

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A	B	C	F	G	H	P	Q	R
0.88939	-0.04820	0.05900	0.15379	-0.58032	0.00000	-0.03497	-0.17522	0.19320
0.88939	-0.08833	0.09913	0.00000	-0.51446	0.26851	-0.03497	-0.24473	0.09020
0.96575	-0.08833	0.02278	0.07640	0.00000	0.25741	-0.05919	-0.24473	0.07652
0.98123	-0.10382	0.02278	0.07586	0.00913	0.00000	-0.08801	-0.23590	0.07652
0.98123	-0.11432	0.03328	0.00000	0.00880	-0.00243	-0.08801	-0.24777	0.01085
0.98125	-0.11432	0.03326	0.00001	0.00000	-0.00243	-0.08796	-0.24777	0.01126
0.98126	-0.11432	0.03326	0.00001	0.00000	0.00000	-0.08768	-0.24786	0.01126
0.98126	-0.11432	0.03326	0.00000	0.00000	0.00000	-0.08768	-0.24786	0.01125
0.98126	-0.11432	0.03326	0.00000	0.00000	0.00000	-0.08768	-0.24786	0.01125
0.98126	-0.11432	0.03326	0.00000	0.00000	0.00000	-0.08768	-0.24786	0.01125
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

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Alpha	Beta	Gamma
-14.06845	27.56112	16.53207
-6.861581	15.46432	-0.2658640
6.3643321E-02	2.1924460E-03	3.7910129E-07
-1.2552463E-11	0.0000000E+00	0.0000000E+00

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ALPTOT	BETTOT	GAMTOT
-20.86638	43.02764	16.26621

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THE ROTATION MATRIX IS :

0.9619108	0.1053204	-0.2522604
-0.2154955	0.8599276	-0.4626946
0.1681945	0.4994319	0.8498697

THE COEFFICIENTS CONSIDERED ARE OF THE 3 X 3 FILTERED IMAGE OF THE AVERAGED CYLINDER.

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THE NUMBER OF ITERATIONS COMPLETED IS : 4

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COEFF. OF X SQUARE TERM IS : 0.6636000
COEFF. OF Y SQUARE TERM IS : 2.0900000E-02
COEFF. OF Z SQUARE TERM IS : -9.2399999E-02
COEFF. OF YZ SQUARE TERM IS : -2.1900000E-02
COEFF. OF XZ SQUARE TERM IS : -0.7604000
COEFF. OF XY SQUARE TERM IS : 0.7727000
COEFF. OF X TERM IS : 0.4242000
COEFF. OF Y TERM IS : -0.2155600
COEFF. OF Z TERM IS : 0.3749000
CONSTANT OF PROP. IS : -0.2534000

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 0.8447758     -0.1276161    -9.2399999E-02 0.3030199     -0.5598051
 0.0000000E+00 0.2925456     -0.3193743     0.3749000
 0.8447758     -0.2625377     4.2521641E-02 0.0000000E+00 -0.4180661
 0.3722935      0.2925456    -0.4878348     6.7580700E-02
 0.8959736     -0.2625377    -8.6761117E-03 8.8566750E-02 0.0000000E+00
 0.3616053      0.2680698    -0.4878348     0.1352357
 0.9235348     -0.2900990    -8.6761117E-03 8.7555319E-02 1.3346768E-02
 0.0000000E+00 0.1914931     -0.5226611     0.1352357
 0.9235348     -0.2967516    -2.0234007E-03 0.0000000E+00 1.3195274E-02
-2.0052318E-03 0.1914931     -0.5370466     5.5175520E-02
 0.9235819     -0.2967516    -2.0704281E-03 1.4292762E-05 0.0000000E+00
-2.0051808E-03 0.1918815     -0.5370466     5.3809207E-02
 0.9235827     -0.2967524    -2.0704281E-03 1.4292757E-05 -1.1742504E-08
 0.0000000E+00 0.1923226     -0.5368887     5.3809207E-02
 0.9235827     -0.2967524    -2.0704281E-03 0.0000000E+00 -1.1742504E-08
 2.8476928E-13 0.1923226     -0.5368900     5.3796187E-02
 0.9235827     -0.2967524    -2.0704281E-03 1.8062406E-21 0.0000000E+00
 2.8476928E-13 0.1923226     -0.5368900     5.3796187E-02
 0.9235827     -0.2967524    -2.0704281E-03 1.8062406E-21 2.1074614E-34
 0.0000000E+00 0.1923226     -0.5368900     5.3796187E-02
 0.0000000E+00 0.0000000E+00 0.0000000E+00 0.0000000E+00 0.0000000E+00
 0.0000000E+00 0.0000000E+00 0.0000000E+00 0.0000000E+00
 0.0000000E+00 0.0000000E+00 0.0000000E+00 0.0000000E+00 0.0000000E+00
 0.0000000E+00 0.0000000E+00 0.0000000E+00 0.0000000E+00
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THE NEW COEFF. OF X SQUARE TERM IS :  0.9235827
THE NEW COEFF. OF Y SQUARE TERM IS :  -0.2967524
THE NEW COEFF. OF Z SQUARE TERM IS :  -2.0704281E-03
THE NEW COEFF. OF X TERM IS :  0.1923226
THE NEW COEFF. OF Y  TERM IS :  -0.5368900
THE NEW COEFF. OF Z TERM IS :  5.3796187E-02
THE NEW CONSTANT OF PROP. IS :  -0.2534000
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   A        B         C         F         G         H         P         Q         R
-----------------------------------------------------  ----------  ----------  ----------  --------------
 0.84478  -0.12762  -0.09240  0.30302  -0.55981  0.00000  0.29255  -0.31937  0.37490
 0.84478  -0.26254   0.04252  0.00000  -0.41807  0.37229  0.29255  -0.48783  0.06758
 0.89597  -0.26254  -0.00868  0.08857   0.00000  0.36161  0.26807  -0.48783  0.13524
 0.92353  -0.29010  -0.00868  0.08756   0.01335  0.00000  0.19149  -0.52266  0.13524
 0.92353  -0.29675  -0.00202  0.00000   0.01320 -0.00201  0.19149  -0.53705  0.05518
 0.92358  -0.29675  -0.00207  0.00001   0.00000 -0.00201  0.19188  -0.53705  0.05381
 0.92358  -0.29675  -0.00207  0.00001   0.00000  0.00000  0.19232  -0.53689  0.05381
 0.92358  -0.29675  -0.00207  0.00000   0.00000  0.00000  0.19232  -0.53689  0.05380
 0.92358  -0.29675  -0.00207  0.00000   0.00000  0.00000  0.19232  -0.53689  0.05380
 0.92358  -0.29675  -0.00207  0.00000   0.00000  0.00000  0.19232  -0.53689  0.05380
 0.00000   0.00000   0.00000  0.00000   0.00000  0.00000  0.00000   0.00000  0.00000
 0.00000   0.00000   0.00000  0.00000   0.00000  0.00000  0.00000   0.00000  0.00000
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   Alpha        Beta        Gamma
-----------------------------------------------------  ----------
 -25.12386      41.68550      13.76233
 -8.667336      8.640893     -0.4083926
 4.7072496E-02  1.3894888E-03  3.6341686E-07
 -6.6850809E-12  0.0000000E+00  0.0000000E+00
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   ALPTOT        BETTOT        GAMTOT
 -33.74412      50.32778      13.35394
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THE ROTATION MATRIX IS :

0.9465714	0.2691452	-0.1776615
-0.3170765	0.6761528	-0.6650413
-5.8866352E-02	0.6858414	0.7253664

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THE COEFFICIENTS CONSIDERED ARE OF THE 5 X 5 FILTERED IMAGE OF THE RAW
CYLINDER.
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THE NUMBER OF ITERATIONS COMPLETED IS :        4
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COEFF. OF X SQUARE TERM IS :  5.7200000E-02
COEFF. OF Y SQUARE TERM IS :  0.5990000
COEFF. OF Z SQUARE TERM IS :  0.4416000
COEFF. OF YZ SQUARE TERM IS : -0.8076000
COEFF. OF XZ SQUARE TERM IS :  0.4595000
COEFF. OF XY SQUARE TERM IS : -0.1490000
COEFF. OF X TERM IS : -0.5915000
COEFF. OF Y  TERM IS :   1.089800
COEFF. OF Z TERM IS :  -1.019800
CONSTANT OF PROP. IS :  0.6643000
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 4.7142603E-02  0.6088773     0.4416000    -0.8618143     0.3400714
 0.0000000E+00 -0.4403836      1.138920     -1.019800
 4.7142603E-02  0.9641879     8.6289465E-02  0.0000000E+00  0.2623782
-0.2163476     -0.4403836     1.527500    -6.2253475E-02
-6.5925181E-02  0.9641879     0.1993573    -0.1412431     0.0000000E+00
-0.1638800     -0.2929416     1.527500    -0.3346617
-7.2402343E-02  0.9706651     0.1993573    -0.1408039    -1.1130215E-02
 0.0000000E+00 -0.1716609      1.545834    -0.3346617
-7.2402343E-02  0.9770384     0.1929839     0.0000000E+00 -1.1084885E-02
 1.0034929E-03 -0.1716609      1.569711    -0.1939274
-7.2518051E-02  0.9770384     0.1930996    -2.0943689E-05  0.0000000E+00
 1.0032743E-03 -0.1756709      1.569711    -0.1903024
-7.2518289E-02  0.9770387     0.1930996    -2.0943688E-05  1.0010066E-08
 0.0000000E+00 -0.1764212      1.569627    -0.1903024
-7.2518289E-02  0.9770387     0.1930996     0.0000000E+00  1.0010066E-08
-1.3371427E-13 -0.1764212      1.569629    -0.1902815
-7.2518289E-02  0.9770387     0.1930996    -2.5195758E-21  0.0000000E+00
-1.3371427E-13 -0.1764212      1.569629    -0.1902815
-7.2518289E-02  0.9770387     0.1930996    -2.5195758E-21 -1.6049783E-34
 0.0000000E+00 -0.1764212      1.569629    -0.1902815
 0.0000000E+00  0.0000000E+00  0.0000000E+00  0.0000000E+00  0.0000000E+00
 0.0000000E+00  0.0000000E+00  0.0000000E+00  0.0000000E+00
 0.0000000E+00  0.0000000E+00  0.0000000E+00  0.0000000E+00  0.0000000E+00
 0.0000000E+00  0.0000000E+00  0.0000000E+00  0.0000000E+00
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THE NEW COEFF. OF X SQUARE TERM IS : -7.2518289E-02
THE NEW COEFF. OF Y SQUARE TERM IS : 0.9770387
THE NEW COEFF. OF Z SQUARE TERM IS : 0.1930996
THE NEW COEFF. OF X TERM IS : -0.1764212
THE NEW COEFF. OF Y TERM IS : 1.569629
THE NEW COEFF. OF Z TERM IS : -0.1902815
THE NEW CONSTANT OF PROP. IS : 0.6643000

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A	B	C	F	G	H	P	Q	R
0.04714	0.60888	0.44160	-0.86181	0.34007	0.00000	-0.44038	1.13892	-1.01980
0.04714	0.96419	0.08629	0.00000	0.26238	-0.21635	-0.44038	1.52750	-0.06225
-0.06593	0.96419	0.19936	-0.14124	0.00000	-0.16388	-0.29294	1.52750	-0.33466
-0.07240	0.97067	0.19936	-0.14080	-0.01113	0.00000	-0.17166	1.54583	-0.33466
-0.07240	0.97704	0.19298	0.00000	-0.01108	0.00100	-0.17166	1.56971	-0.19393
-0.07252	0.97704	0.19310	-0.00002	0.00000	0.00100	-0.17567	1.56971	-0.19030
-0.07252	0.97704	0.19310	-0.00002	0.00000	0.00000	-0.17642	1.56963	-0.19030
-0.07252	0.97704	0.19310	0.00000	0.00000	0.00000	-0.17642	1.56963	-0.19028
-0.07252	0.97704	0.19310	0.00000	0.00000	0.00000	-0.17642	1.56963	-0.19028
-0.07252	0.97704	0.19310	0.00000	0.00000	0.00000	-0.17642	1.56963	-0.19028
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

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Alpha	Beta	Gamma
-7.688372	39.50777	40.75703
-4.519698	5.172774	-1.195895
2.7384600E-02	7.6535594E-04	1.0796233E-06
-3.6497606E-12	0.0000000E+00	0.0000000E+00

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ALPTOT	BETTOT	GAMTOT
-12.18069	44.68131	39.56113

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THE ROTATION MATRIX IS :

0.8062407	-0.3102598	-0.5037009
-0.1032203	0.7646025	-0.6361828
0.5825129	0.5649087	0.5844286